Mathematical Finance Final Cheat Sheet

Stochastic Processes in Continuous Time

- $X = (X_t)_{t \geq 0}$ **adapted**: r.v. X_t is $\mathcal{F}_t$-measurable $\forall t > 0$
- X adapted, $E(|X_t|) < \infty \quad \forall t > 0$; for $t > s$,
  submartingale: $E(X_t|\mathcal{F}_s) \geq X_s$
  supermartingale: $E(X_t|\mathcal{F}_s) \leq X_s$
  martingale: $E(X_t|\mathcal{F}_s) = X_s$
- **Markov process**: X adapted, for all $t, s > 0$ and bounded functions $f : \mathbb{R} \to \mathbb{R}$,
  $E(f(X_{t+s})|\mathcal{F}_t) = E(f(X_{t+s})|\sigma(X_t))$
- R.v. τ is a **stopping time** w.r.t. $\mathcal{F}_t$ if for all $t \geq 0$ $\{\tau \leq t\} \in \mathcal{F}_t$
  σ-field of events observable up to τ:
  $\mathcal{F}_\tau := \{A \in \mathcal{F} : A \cap \{\tau \leq t\} \in \mathcal{F}_t \quad \forall t \geq 0\}$
- $X = (X_t)_{t \geq 0}$ on $(\sigma, \mathcal{F}, P)$ is standard one-dimensional **Brownian motion**, if
  i. $X_0 = 0$ a.s.
  ii. $\forall t, u \geq 0$ $X_{t+u} - X_t$ is indep. of X_s $\forall s \leq t$
  iii. $X_{t+u} - X_t \sim N(0, u)$
  iv. X has continuous sample paths
- Function $X : [0, \overline{T}] \to \mathbb{R}$
  First variation:
  $\mathrm{Var}(X) := \sup \left\{ \sum_{t_i \in \tau} |X(t_i) - X(t_{i-1})|, \tau \text{ a partition of } [0, \overline{T}] \right\}$
  If $\mathrm{Var}(X) < \infty$, X is of finite variation
  Quadratic variation: $(\tau_n)_{n \in \mathbb{N}} \ldots$ sequence of partitions of $[0, \overline{T}]$ s.t. $|\tau_n| \to 0$ as $n \to \infty$;
  $V_t^2(X; \tau_n) := \sum_{t_i \in \tau_n; t_i < t} (X(t_i) - X(t_{i-1}))^2$
- X continuous with quadr. var. $[X]_t$, $A : [0, \overline{T}] \to \mathbb{R}$ continuous function of first variation, $Y_t := X_t + A_t$; then $[Y]_t = [X]_t$

Pathwise Itô Calculus

Consider continuous function $X : [0, T] \to \mathbb{R}$ with continuous quadratic variation $[X]_t$

- **Itô's formula**: Consider twice continuously differentiable function $F : \mathbb{R} \to \mathbb{R}$; for $t \leq \overline{T}$
  $F(X_t) = F(X_0) + \int_0^t F'(X_s) dX_s + \frac{1}{2} \int_0^t F''(X_s) d[X]_s$,
  $$\int_0^t F'(X_s) dX_s := \lim_{n \to \infty} \sum_{t_i \in \tau_n; t_i \leq t} F'(X_{t_{i-1}})(X_{t_i} - X_{t_{i-1}})$$
- Let $F \in \mathcal{C}^1(\mathbb{R})$; then the function $t \to F(X_t)$ has quadratic variation $\int_0^t (F'(X_s))^2 d[X]_s$
  Corollary: For $f \in \mathcal{C}^1(\mathbb{R})$ $I_t := \int_0^t f(X_s) dX_s$ is well-defined and $[I]_t = \int_0^t f^2(X_s) d[X]_s$
- Local martingale M with continuous trajectories and continuous quadratic variation, $f \in \mathcal{C}^1(\mathbb{R})$; then $I_t(\omega) = \int_0^t f(M_s(\omega)) dM_s(\omega)$ is a local martingale

Pathwise Itô Calculus (cont.)

- **Semimartingale**: $X_t = X_0 + M_t + A_t$, M local martingale, A adapted process with continuous trajectories of finite variation
- **Covariation**:
  $[X, Y]_t := \lim_{n \to \infty} \sum_{t_i \in \tau_n, t_i \leq t} (X_{t_i} - X_{t_{i-1}})(Y_{t_i} - Y_{t_{i-1}})$
  $[X, Y]_t$ exists iff $[X + Y]_t$ exists
  $[X, Y]_t = \frac{1}{2} ([X + Y]_t - [X]_t - [Y]_t)$
- **d-dimensional Itô**:
  Given continuous functions $X = (X^1, \ldots, X^d) : [0, \overline{T}] \to \mathbb{R}$ with continuous covariation and a twice continuously differentiable function $F : \mathbb{R}^d \to \mathbb{R}^d$,
  $$F(X_t) = F(X_0) + \sum_{i=1}^d \int_0^t \frac{\partial}{\partial x_i} F(X_s) dX_s^i + \frac{1}{2} \sum_{i,j=1}^d \int_0^t \frac{\partial^2}{\partial x_i \partial x_j} F(X_s) d[X^i, X^j]_s$$
- **Itô's product formula**:
  $X_t Y_t = X_0 Y_0 + \int_0^t X_s dY_s + \int_0^t Y_s dX_s + [X, Y]_t$
- **Itô for time dependent functions**:
  $$F(t, X_t) = F(0, X_0) + \int_0^t F_t(s, X_s) ds + \int_0^t F_x(s, X_s) dX_s + \frac{1}{2} \int_0^t F_{xx}(s, X_s) d[X]_s$$

Itô Processes and Feynman-Kac

- **Itô process** satisfies the SDE
  $X_t = x + \int_{t_0}^t \mu(s, X_s) ds + \int_{t_0}^t \sigma(s, X_s) dW_s$
  Short: $dX_t = \underbrace{\mu(t, X_t) dt}_{\text{finite variation part}} + \underbrace{\sigma(t, X_t) dW_t}_{\text{martingale part}}$
  (semimartingale)
  $[X]_t = \int_{t_0}^t \sigma^2(s, X_s) ds$
- Examples (with $t_0 = 0$):
  Arithmetic Brownian motion:
  $\mu(t, X_t) = \mu, \quad \sigma(t, X_t) = \sigma; \quad X_t = X_0 + \mu t + \sigma W_t$
  Geometric Brownian motion:
  $\mu(t, X_t) = \mu X_t$, $\sigma(t, X_t) = \sigma X_t$; $X_t = X_0 e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma W_t}$
  Ornstein Uhlenbeck process:
  $\mu(t, X_t) = \kappa(\theta - X_t)$, $\sigma(t, X_t) = \sigma$, $X_0 = \bar{r}$;
  $X_t = \bar{r} e^{-\kappa t} + \theta(1 - e^{-\kappa t}) + \sigma e^{-\kappa t} \int_0^t e^{\kappa s} dW_s$
- **Feynman Kac formula**:
  (Links stochastic processes & parabolic PDEs)
  Consider terminal value problem
  $\frac{\partial F}{\partial t}(t, x) + \mu(x) \frac{\partial F}{\partial x}(x) + \frac{1}{2} \sigma^2(x) \frac{\partial^2 F}{\partial x^2}(x) = r(x) F(t, x)$,
  $F(T, x) = \phi(x)$
  Representation in terms of some Itô process with $dX_t = \mu(X_t) dt + \sigma(X_t) dW_t$:
  $F(t_0, X_{t_0}) = E_{X_{t_0}} \left(exp \left(-\int_0^{T - t_0} r(X_s) ds \right) \phi(X_{T - t_0}) \right)$

Black Scholes Model

- Money market account $B_t = e^{rt}$
- Asset price dynamics $dS_t = \mu S_t dt + \sigma S_t dW_t$ (GBM)
- **Terminal value claim** with payoff $H = h(S_T)$
- $\phi(t, S_t)$ number of stocks, $\eta(t, S_t)$ units of money market account in the portfolio
- Portfolio value $V(t, S_t) = S_t \phi(t, S_t) + B_t \eta(t, S_t)$
- Gains from trade $G_t = \int_0^t \phi(s, S_s) dS_s + \int_0^t \eta(s, S_S) dB_s$
- **Selffinancing strategy**: $V(t, S_t) = V(0, S_0) + G_t$
- **Replicating strategy** if $V(T, S) = h(S)$ for all $S > 0$
- **Black-Scholes PDE** (representation of fair value as terminal value problem): If $V : [0, T] \times \mathbb{R}^+ \to \mathbb{R}$ solves $V_t(t, S) + rSV_s(t, S) + \frac{1}{2} \sigma^2 S^2 V_{ss}(t, S) = rV(t, S)$ with terminal condition $V(T, S) = h(S)$, then the strategy replicates the terminal value claim and $V(t, S_t)$ is the fair price at t
- **Black-Scholes formula** (solving terminal value problem, using Feynman Kac):
  $C_{BS}(t, S; \sigma, r, K, T) := SN(d_1) - e^{-r(T-t)} KN(d_2)$,
  $d_1 = \frac{ln(\frac{S}{K}) + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma \sqrt{T-t}}, \quad d_2 = d_1 - \sigma \sqrt{T - t}$
  No-arbitrage price of a European call with strike K, time to maturity T, volatility σ and interest rate r
- **Put-Call parity** $C_t + e^{-r(T-t)} K = S_t + P_t$
  $C_t / P_t \ldots$ European call/put price
- **Black-Scholes price for European put**:
  $P_{BS}(t, S; \sigma, r, K, T) = -S_t N(-d_1) + K e^{-r(T-t)} N(-d_2)$
- **Greeks**:
  $\Delta_C = \frac{\partial C}{\partial S} = N(d_1), \quad \Delta_P = \frac{\partial P}{\partial S} = \Delta_C - 1 = -N(-d_1)$
  $\Gamma_C = \frac{\partial^2 C}{\partial S^2}, \quad \Gamma_C = \Gamma_P = \frac{\varphi(d_1)}{S_t \sigma \sqrt{T-t}}$
  $\mathrm{Vega}_C = \frac{\partial C}{\partial \sigma}, \quad \mathrm{Vega}_C = \mathrm{Vega}_P = S_t \varphi(d_1) \sqrt{T - t}$
  $\Theta_C = \frac{\partial C}{\partial t} = -\frac{S_t \sigma \varphi(d_1)}{2\sqrt{T-t}} - rKe^{-r(T-t)} N(d_2)$,
  $\Theta_P = \frac{\partial P}{\partial t} = -\frac{S_t \sigma \varphi(d_1)}{2\sqrt{T-t}} + rKe^{-r(T-t)} N(-d_2)$
  $\rho_C = \frac{\partial C}{\partial r} = K(T - t) e^{-r(T-t)} N(d_2)$,
  $\rho_P = \frac{\partial P}{\partial r} = -K(T - t) e^{-r(T-t)} N(-d_2)$
- Tracking error $e_T = \frac{1}{2} \int_0^T S_t^2 (\sigma_t^2 - (\sigma^*)^2) H_{SS}^{BS}(t, S_t) dt$

Other

- Density of the normal distr. $f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$
- $E(1_A) = P(A)$
- Lognormal rv $X \sim LN(\mu, \sigma^2)$, $E(X) = \exp(\mu + \frac{1}{2}\sigma^2)$

Good luck!