Dr. Christa Cuchiero
Institute for Statistics and Mathematics
WU Vienna
christa.cuchiero@wu.ac.at

Master QFin, CTFI

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1. Stochastic processes (4 points)
Consider a compound Poisson process

$$S_t = \sum_{i=1}^{N_t} Z_i,$$

where N is a Poisson process with parameter λ, independent of $(Z_i)_{i\in\mathbb{N}}$ which is a sequence of independent standard Gaussian random variables.

a) Determine the set of possible jump sizes.

b) Define the waiting time until the first jump as

$$T_1 = \inf\{t > 0 \,|\, S_t \neq 0\}.$$

Determine the distribution of T_1.

c) Compute the moment generating function of S, i.e. $E[e^{uS_t}]$ for $u \in \mathbb{R}$.

2. Quadratic covariation (4 points)

a) Let $(M_t)_{t\geq 0}$ and $(N_t)_{t\geq 0}$ are continuous local martingales and let $([M,N]_t)_{t\geq 0}$ denote their continuous covariation process along a fixed sequence of partitions. Show that

$$(MN - [M,N])_{t\geq 0}$$

is a local martingale.

b) Let $W_1(t)$ und $W_2(t)$ be independent Brownian motions and $t \mapsto \rho(t)$ a C^1-function with values in $]-1,1[$. Define a new Brownian motion W_3 by

$$W_3(t) := \int_0^t \rho(s)dW_1(s) + \int_0^t \sqrt{1-\rho^2(s)}dW_2(s).$$

Compute the covariations $[W_1, W_3]$ and $[W_2, W_3]$.

3. Ito calculus (3 points)
Let B be a Brownian motion.

a) Use Ito's formula to show or disprove that $(B_t^3)_{t\geq 0}$ is a martingale.

b) Show, using Itô's formula, that the following process is a local martingale

$$X_t = (B_t + t)\exp(-B_t - \frac{1}{2}t).$$

4. Black Scholes model (9 points)

a) Compute the limit of the Black-Scholes formula for $\sigma \to 0$ and $\sigma \to \infty$ and give an economic interpretation.

b) Consider a *gap call* with payoff

$$H(S_T) = (S_T - L)1_{\{S_T > K\}},$$

where $K, L \geq 0$.

(i) Draw the payoff function. For which values of K and L does it take negative values?

(ii) Find the price in the Black-Scholes model by using the risk neutral evaluation formula and splitting the option in two digital options (i.e., in an asset-or-nothing option with payoff $S_T 1_{\{S_T > C\}}$ and a cash-or-nothing option ~~with~~ where payoff is $M 1_{\{S_T > N\}}$, where the constants C, M, N have to be determined).

(iii) What is the Delta of the option?

(iv) What is the amount that has to be invested in the bank account for perfect replication of the option. (The formula without using the concrete values is enough here).