CTF I

Final Preparation Lecture

Theresa Traxler

Vienna University of Economics and Business

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Overview

1. **Recap**

2. **Exercises**

Feynman Kac I

Consider a PDE of the form

$$\frac{\partial f}{\partial t}(t,x) + \mu(x)\frac{\partial f}{\partial x}(x) + \frac{1}{2}\sigma^2(x)\frac{\partial^2 f}{\partial x^2}(x) = r(x)f(t,x), \qquad f(T,x) = \Phi(x),$$

and an Itô process with the following dynamics

$$dX_t = \mu(X_t)dt + \sigma(X_t)dW_t, \qquad X_{t_0} = x.$$

Then,

$$\begin{aligned}
f(t_0, X_{t_0}) &= \mathbb{E}\left(\exp\left(-\int_{t_0}^{T} r(X_s)ds\right)\Phi(X_T)\middle|\; \underbrace{\mathcal{F}_{t_0}}_{\text{all info up to } t_0}\right) \\
&\overset{\text{Markov property}}{=} \mathbb{E}\left(\exp\left(-\int_{t_0}^{T} r(X_s)ds\right)\Phi(X_T)\middle|\; \underbrace{X_{t_0} = x}_{\text{only starting value}}\right) \\
&= \mathbb{E}_x\left(\exp\left(-\int_{0}^{T-t_0} r(X_s)ds\right)\Phi(X_{T-t_0})\right).
\end{aligned}$$

Feynman Kac II

- We can use the Markov property (only present value is relevant for the expectation, past values do not give any additional information), because we know X_t is an Itô process.
- In the last step we intuitively use that if $\mu(X_t)$ and $\sigma(X_t)$ do not depend on time, which is the case we consider, it does not matter whether starting at t_0 we look forward to T, or from 0 we look to $T - t_0$.
- $\mathbb{E}_x$ is just another way of writing the conditional expectation.

It can be used in two ways:

- We are given a PDE and want to solve it, i.e. finding out what the f is. To do so, we identify from the given PDE $\mu(x), \sigma(x), r(x)$ and $\Phi(x)$ just by looking at the given problem and comparing with the general form. Having done so, we can determine our Itô process by plugging in the information we got into the dynamics $dX_t = \mu(X_t)dt + \sigma(X_t)dW_t$. Now we can determine X_{T-t_0} and plug it into the expectation as $\Phi(X_{T-t_0})$ together with $r(X_s)$, which allows us to find the desired result, an expression for f.
- We are given that some function can be expressed as an expectation and the dynamics of the Itô process used. We want to derive the corresponding PDE, for which the function is a solution. This means that, by looking at the given expectation, we can determine $r(x)$ and $\Phi(x)$. From the dynamics of the Itô process we can determine $\mu(x)$ and $\sigma^2(x)$. Plugging these terms into the general form yields the corresponding PDE.

Black Scholes Model - Key Takeaways I

- Two traded assets:
  $B_t = e^{rt}$ (riskless, money market account, deterministic)
  $S_t = S_0 e^{\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma W_t}$, $dS_t = \mu S_t dt + \sigma S_t dW_t$ (risky, stock, GBM)
- This implies that log-returns are normally distributed with constant drift and volatility and independent, which is not true in real life (e.g. volatility clustering, correlation of returns, heavy tails)
- We want to price **terminal value claims** whose payoff is of the form $H = h(S_T)$ (maturity T)
- Consider a portfolio with $\phi(t, S_t)$ stocks and $\eta(t, S_t)$ units of the money market account at t (**Markov trading strategy**)
- **Selffinancing trading strategy:** $V(t, S_t) = V(0, S_0) + G_t$ (i.e. the portfolio is rearranged using only the initial wealth and the gains made up to t and one cannot put extra money in)
  Note: The gains from trade can be better understood as the limit of the discrete time case where the investor decides at t_{i-1} about the allocation (the number of stocks) and keeps it like that until t_i (predictable process). The gain between two time points is then the difference of the portfolio values at these time points and the whole gain is obtained by summing up all the small gains, i.e. $G^n_{t_i} = \sum_{j=1}^{i} \left(\phi^n_{t_j}(S_{t_j} - S_{t_{j-1}}) + \eta^n_{t_j}(B_{t_j} - B_{t_{j-1}}) \right), t_i \in \tau_n$.
  Using the Itô integral definition, the limit is $G_t = \int_0^t \phi(s, S_s) dS_s + \int_0^t \eta(s, S_s) dB_s$.
- A **replicating strategy** means that we can create a selffinancing portfolio with a certain portion of risky and riskless assets such that its terminal value is the same as the payoff of some terminal value claim. If the value at maturity is the same, it should also be the same before maturity, meaning if we know the value of the replicating portfolio at t, we can use this information to price the claim at t.

Black Scholes Model - Key Takeaways II

Black Scholes PDE

$V_t(t,S) + rSV_S(t,S) + \frac{1}{2}\sigma^2 S^2 V_{SS}(t,S) = rV(t,S)$

- A hedging strategy with stock position $\phi(t,S) = V_S(t,S)$ and value $V(t,S)$ is **selffinancing**, because the proof of the Black Scholes PDE is actually based on using the time dependent Itô formula for the portfolio value, using the selffinancing condition and then comparing the terms. So it is clear that if V solves the PDE, it is selffinancing. If additionally $V(T,S) = h(S)$, then we have a **replicating strategy** of the terminal value claim. The value of the corresponding cash position in the hedging portfolio can be obtained by $V - V_S \cdot S$ and the number is $\eta = (V - V_S \cdot S)e^{-rt}$ (dividing by value of one riskless asset).
- So the value of the replicating portfolio of a terminal value claim is described by the BS PDE with terminal condition $V(T,S) = h(S)$ (terminal value problem).

Risk Neutral Pricing

$V(t,S) = \mathbb{E}_S\left(e^{-r(T-t)}h(S_{T-t})\right)$

- For $dS_t = rS_t dt + \sigma S_t dW_t$, using FK gives the risk neutral pricing formula. The value at time t is the expected value of the discounted terminal payoff, assuming the underlying stockprice follows GBM with drift r.
- Call and put options have payoffs $(S-K)^+$ and $(K-S)^+$.

  Call price $C_{BS} = (t, S; \sigma, r, K, T) = SN(d_1) - e^{-r(T-t)}KN(d_2)$; $d_1 = \frac{\ln\frac{S}{K} + \left(r + \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}$, $d_2 = d_1 - \sigma\sqrt{T-t}$

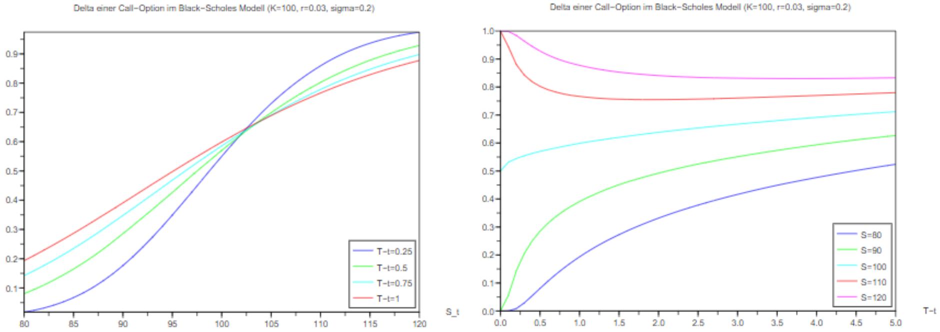
  Put price $P_{BS} = -SN(-d_1) + Ke^{r(T-t)}N(-d_2)$

Risk Neutral Pricing: Why is the drift r?

- Theory tells us that options with the same volatility and different drift have the same value, i.e. the option price is independent from the drift, which is not intuitive (buy and hold investor: choose the option with higher drift as on average it has a higher payoff). However, it is reasonable because in the model we consider the price of the option is derived based on a replicating portfolio consisting of the underlying stock and a risk free asset (we assume there exists a portfolio perfectly replicating the claim which is actually not true). If μ is high i.e. you make money with the call option, also the replicating portfolio will be profitable. On the other hand, if the portfolio won't be profitable, also the option will be out of the money in this case. So the drift does not matter as it impacts the value of the underlying and the performance of the hedging strategy in the same way and the two cancel. The cost of implementing the hedge is the same no matter what μ is, it only depends on the volatility.
- Does this mean now that in general we always assume $\mu = r$? No, for pricing we do not consider the real model with μ, but an "artificial" one with drift r and where the discounted price process follows a martingale. This property is needed because only then the price at time t exactly equals the expectation of the discounted payoff!
- Think of probability measures. Under the real world measure (physical measure) the drift is μ but we can find another measure which makes the drift become r, which is then called equivalent martingale measure. This can be done using the Girsanov theorem from CTF II. For pricing, to be precise, we actually use the expectation with respect to this martingale measure (or risk-neutral measure). A risk neutral measure exists iff the market is arbitrage-free and it is unique iff the market is complete.
- What that means is we keep the actual problem as given with drift μ, but just for pricing we use a setup where it is changed to r such that it fits to the BS PDE, because the actual drift is "not important" in this context anyway. However μ is still important and counts for solving other problems like portfolio optimization.
- Why is it called risk neutral? In this setup the fair price is computed as the expectation of the discounted terminal value, which means you do not include any unique factors that drive the respective asset up or down, i.e. no risks considered.

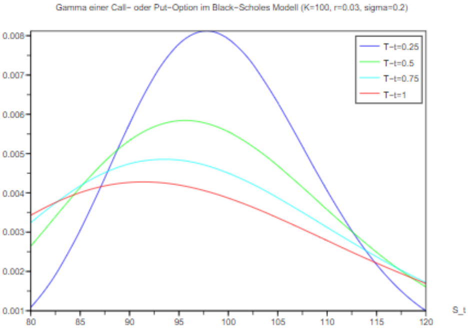
Greeks I

- **Delta:** First derivative w.r.t. the underlying, determines the number of stocks in a Delta hedging portfolio (= replicating portfolio used to hedge);
  $\Delta_C = \frac{\partial C_{BS}}{\partial S} = N(d_1)$, $\Delta_P = \frac{\partial P}{\partial S} = -N(-d_1)$
  For a call option, a higher value of the underlying means a higher option value by definition of the payoff. Therefore it is intuitive that the Delta of the call increases in S. Considering a call and the time to maturity $T - t$ going to 0, the Delta basically converges to $1_{S>K}$, i.e. if the value of the underlying is bigger than the strike price (call option is in the money) shortly before maturity, an investor would allocate more to the stock and the other way round; only for the option being at the money it tends to 0.5



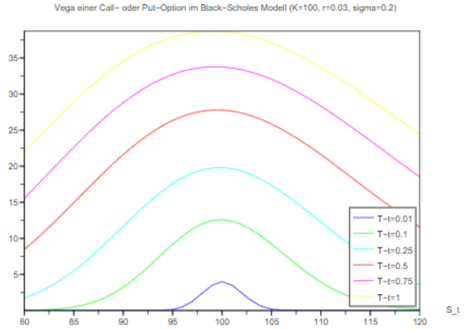
Greeks II

- **Gamma:** Second derivative w.r.t. the underlying, measures how fast the Delta changes when the underlying changes and hence how often the hedging portfolio needs to be rebalanced such that it stays a Delta neutral strategy (it is the slope w.r.t. S of the Delta); a lower Gamma is better in the sense that if the value of the underlying changes, the shares to hold in the replicating portfolio does not change a lot and less trades are needed than for a high Gamma; it is always positive for put and call
  $\Gamma_C = \frac{\partial^2 C}{\partial S^2}$, $\Gamma_C = \Gamma_P = \frac{\varphi(d_1)}{S_t \sigma \sqrt{T-t}}$ (intuitive that it's the same as the Delta differs only by a constant)
  When the value of the underlying is close to the strike price of an option, "the picture is not so clear" and more rebalancing is needed. This means the Gamma is biggest for S close to K.
  For a smaller time to maturity $T-t$ fast rebalancing might be needed and therefore the Gamma is more peaked the smaller it is.
  Options at the money with small time to maturity are difficult to hedge, as the Delta is almost an indicator function for small $T-t$ and so the Delta can basically jump from 0 to 1, i.e. it changes a lot, so the Gamma is biggest for this case.



Greeks III

- **Vega:** First derivative w.r.t. the volatility (how big is the change of the option value in case the volatility is changed by one unit); sensitive to volatility misspecifications (volatility is the hardest parameter to determine); it is always positive
  $Vega_C = \frac{\partial C}{\partial \sigma}$, $Vega_C = Vega_P = S_t \varphi(d_1)\sqrt{T-t}$
  It is increasing in time to maturity so for small $T-t$ a difference in volatility doesn't matter a lot. It is highest if S is close to K because again the picture is not so clear and volatility can have a bigger impact.
  The Vega is important because the volatility in reality is not constant and a high Vega means that the value of the option is very sensitive to changes in this unknown and hard to determine parameter. Then there is a lot of model risk (sensitive to misspecification - value changes a lot if the volatility is chosen wrongly).



Delta hedging (constructing a Delta neutral strategy)

- A Delta neutral strategy means selling a contingent claim and additionally having a hedging portfolio consisting of risky and riskless assets that replicates the claim in a way such that the Deltas "compensate" each other and all in all, the total value remains unchanged when a small change in the value of the underlying S occurs.
- The delta neutral strategy is determined by the option Delta; Why? The Delta describes how the value of the option changes when S is changed by one unit. If we want to hedge, we need to reflect exactly this change in the hedging portfolio, meaning we need Delta units of S in this portfolio such that both the change in the claim and in the portfolio are equally big.
- $\Delta_C = \frac{\partial C_{BS}}{\partial S} = N(d_1)$ units of S and $(C - \Delta_C S)\, e^{-rt} = e^{-rT} K N(d_2)$ units of the money market account in the portfolio for delta neutral hedging of a call
- $\Delta_P = \frac{\partial P}{\partial S} = - \underbrace{N(-d_1)}_{=1-N(d_1)}$ units of S and $(P_{BS} - \Delta_P S)\, e^{-rt} = e^{-rT} K N(-d_2)$ units of the money market account in the portfolio for delta neutral hedging of a put
- But in reality this is not a perfect hedge as markets do not follow the Black Scholes model (i.e. stochastic volatility). There is an error.

Final Exam 2021 I

Feynman Kac. Consider the solution X of the SDE

$$dX_t = (\theta - X_t)dt + \sigma\sqrt{X_t(1 - X_t)}dW_t$$

for parameters $\sigma > 0$ and $\theta \in (0, 1)$. (It is known that a unique solution exists for every initial value $0 \leq 1$ and that the solution is an element of $[0, 1]$ for all t.) Define for parameters $\rho_0, \rho_1 > 0$ the function

$$F(t, x) = E_x\left(\exp\left(-\int_0^{T-t} \rho_0 + \rho_1 X_s ds\right)\right), (t, x) \in [0, T] \times [0, 1].$$

Use the Feynman Kac formula to derive the terminal value proplem for F.

- By looking at the given expectation, we can determine $r(x) = \rho_0 + \rho_1 x$ and $\Phi(x) = 1$.
- From the dynamics of the Itô process we can determine $\mu(x) = \theta - x$ and $\sigma(x) = \sigma\sqrt{x(1 - x)}$.
- Plugging these terms into the general form yields the corresponding PDE,

$$\frac{\partial F}{\partial t}(t, x) + (\theta - x)\frac{\partial F}{\partial x}(x) + \frac{1}{2}\sigma^2 x(1 - x)\frac{\partial^2 F}{\partial x^2}(x) = (\rho_0 + \rho_1 x)F(t, x), \quad F(T, x) = 1.$$

Final Exam 2021 II

Uniqueness of semimartingale decomposition. Consider a continuous semimartingale

$$X_t = X_0 + M_t + A_t, t \geq 0,$$

where M is a local martingale with continuous paths and where $A_t = \int_0^t a_s ds$ is a process of finite variation. Show that this decomposition is unique, that is for any other decomposition $X_t = X_0 + \widetilde{M}_t + \widetilde{A}_t$ with these properties it holds that $M = \widetilde{M}$ and $A = \widetilde{A}$.

$$X_t = X_0 + M_t + A_t = X_0 + \widetilde{M}_t + \widetilde{A}_t \implies M_t - \widetilde{M}_t = \widetilde{A}_t - A_t$$

The right hand side is an adapted process of finite variation, and the left hand side is a local martingale with continuous trajectories. Therefore we can use Proposition 3.11 (handout p.28) for local martingales with continuous trajectories of finite variation to get $M_t - \widetilde{M}_t = M_0 - \widetilde{M}_0$ almost surely.
For $t = 0$,

$$M_0 - \widetilde{M}_0 = \widetilde{A}_0 - A_0 \overset{A_0 = \widetilde{A}_0 = 0}{\implies} M_0 - \widetilde{M}_0 = 0$$

Therefore, $M = \widetilde{M}$ a.s. and $A = \widetilde{A}$ a.s., so the decomposition is unique. $\square$

Final Exam 2021 III

Black Scholes model and binary option. Consider in the context of the Black Scholes model with stock price dynamics $dS_t = \mu S_t dt + \sigma S_t dW_t$, initial stock price $S_0 > 0$ and with money market account $B_t = \exp(rt)$ for $r > 0$ a so-called binary put option with payoff $h(S_T) = 1_{\{S_T \leq K\}}$ for some $K > 0$. Use the risk neutral pricing formula to compute the price of the option at $t = 0$ and the stock position of the corresponding hedging portfolio.

"In the context of the Black Scholes model" means that the BS PDE holds, i.e. we consider the following terminal value problem:

$$V_t + rSV_S + \frac{1}{2}\sigma^2 S^2 V_{SS} = rV, \quad V(T, S) = h(S_T) = 1_{\{S_T \leq K\}}$$

The fair price of the terminal value claim is the value V of the replicating strategy at time t. The risk neutral pricing formula is obtained via FK:
$\mu(S) = rS$, $\sigma^2(S) = \sigma^2 S^2$, $r(S) = r$, $\Phi(S) = 1_{\{S_T \leq K\}}$

$$V(t, S) = \mathbb{E}_{S_0 = S}\left(\exp\left(-\int_0^{T-t} r\,ds\right) 1_{\{S_{T-t} \leq K\}}\right),$$

where S is a GBM with drift r (because we are in the risk neutral pricing setup!) and initial value S_0,
$S_s = rS_s ds + \sigma S_s dW_s \implies S_s = S_0 e^{\left(r - \frac{\sigma^2}{2}\right)s + \sigma W_s}$ and $S_{T-t} = S_0 e^{\left(r - \frac{\sigma^2}{2}\right)(T-t) + \sigma W_{T-t}}$.
Equipped with this probabilistic expression for $V(t, S)$, we can now compute the price of the option at t.

Final Exam 2021 IV

$$V(t,S) = \mathbb{E}_S\left(\underbrace{\exp\left(-\int_0^{T-t} r ds\right)}_{=e^{-r(T-t)},\text{ deterministic}} 1_{\{S_{T-t}\leq K\}}\right) = e^{-r(T-t)}\mathbb{E}_S\left(1_{\{S_{T-t}\leq K\}}\right) = e^{-r(T-t)}\mathbb{P}\left(S_{T-t}\leq K\right)$$

$$= e^{-r(T-t)}\mathbb{P}\left(Se^{\left(r-\frac{\sigma^2}{2}\right)(T-t)+\sigma W_{T-t}} \leq K\right) = e^{-r(T-t)}\mathbb{P}\left(\frac{S}{K}e^{\left(r-\frac{\sigma^2}{2}\right)(T-t)+\sigma W_{T-t}} \leq 1\right)$$

$$= e^{-r(T-t)}\mathbb{P}\left(\ln(S) - ln(K) + \left(r - \frac{\sigma^2}{2}\right)(T-t) + \sigma W_{T-t} \leq 0\right)$$

$$= e^{-r(T-t)}\mathbb{P}\left(W_{T-t} \leq \frac{\ln(K) - \ln(S) - \left(r - \frac{\sigma^2}{2}\right)(T-t)}{\sigma}\right)$$

$$= e^{-r(T-t)}\mathbb{P}\left(\underbrace{\frac{W_{T-t}}{\sqrt{T-t}}}_{\sim\mathcal{N}(0,1)} \leq \frac{ln(K) - \ln(S) - \left(r - \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}\right)$$

Final Exam 2021 V

$$V(t,S) = e^{-r(T-t)} N\left(\frac{\ln(K) - \ln(S) - \left(r - \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}\right) =: e^{-r(T-t)} N(d)$$

N is the cdf of the standard normal distribution. Now we can compute the stock position for the corresponding hedging portfolio.

$$\Delta = \frac{\partial V(t,S)}{\partial S} = e^{-r(T-t)} \frac{\partial N(d)}{\partial d} \frac{\partial d}{\partial S} = -e^{-r(T-t)} \varphi(d) \frac{1}{S\sigma\sqrt{T-t}}$$

$$\implies V(0,S_0) = e^{-rT} N\left(\frac{\ln(K) - \ln(S_0) - \left(r - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right)$$

$$\Delta = -e^{-rT} \varphi\left(\frac{\ln(K) - \ln(S_0) - \left(r - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right) \frac{1}{S_0\sigma\sqrt{T}}$$

φ is the density of the standard normal distribution.

Final Exam 2021 VI

Volatility skew and Black Scholes model. Briefly explain what is meant by the skew pattern of implied volatility (or implied volatility skew) on options' markets. Discuss how the implied volatility skew is related to empirical deficiencies of the Black Scholes model.

- **Implied volatility:** Use estimated σ embedded in actually traded options to find what market "thinks" about volatility
- If the market would believe BS was true, the implied volatility would be constant, independent of strike price K and time to maturity $T - t$
- Reality: implied volatilities display typical pattern = skew as market knows that BS is not true
- **Skew** therefore represents deviations of reality from BS (volatility clustering, heavy, tails, jumps, etc.)

Some Old Homework Exercises I

Logarithmic stock price. Consider in the context of the Black Scholes model with parameters μ, σ, r a terminal value claim with payoff $h(S_T) = \ln S_T$

- Use the risk-neutral pricing formula to compute the price of this option
  For pricing we assume $\mu = r$, so $dS_t = rS_t dt + \sigma S_t dW_t$ and $S_{T-t} = S_0 e^{\left(r-\frac{1}{2}\sigma^2\right)(T-t)+\sigma W_{T-t}}$

$$
\begin{aligned}
V(t,S) &= \mathbb{E}_S\left(e^{-r(T-t)}h(S_{T-t})\right) = e^{-r(T-t)}\mathbb{E}_S\left(\ln(S_{T-t})\right) \\
&= e^{-r(T-t)}\left[\ln S + \left(r - \frac{1}{2}\sigma^2\right)(T-t) + \sigma \underbrace{\mathbb{E}_S\left(W_{T-t}\right)}_{=0 \text{ (BM)}}\right] \\
&= e^{-r(T-t)}\left[\ln S + \left(r - \frac{1}{2}\sigma^2\right)(T-t)\right]
\end{aligned}
$$

Some Old Homework Exercises II

- Give a selffinancing replicating strategy for the claim

  Stock position $V_S(t,S)$:

$$\phi(t,S) = V_S(t,S) = e^{-r(T-t)}\frac{1}{S}$$

Units of riskless asset:

$$\eta(t,S) = (V(t,S) - V_S(t,S)S)e^{-rt} = e^{-r(T-t)}\left[\ln S + \left(r - \frac{1}{2}\sigma^2\right)(T-t) - 1\right]e^{-rt}$$
$$= e^{-rT}\left[\ln S + \left(r - \frac{1}{2}\sigma^2\right)(T-t) - 1\right]$$

Some Old Homework Exercises III

Discounting. Consider a model where the stock price follows geometric Brownian motion, $dS_t = \mu S_t dt + \sigma S_t dW_t$. Denote the money market account by $B_t = e^{rt}$. Compute the dynamics of the discounted stock price $\tilde{S}_t = S_t / B_t$ using Itô's product formula. For which value of μ is $\tilde{S}_t$ a martingale?
Itô's product formula (lecture notes p. 30, Corollary 3.16) is

$$X_t Y_t = X_0 Y_0 + \int_0^t X_s dY_s + \int_0^t Y_s dX_s + [X, Y]_t,$$

$$d(XY)_t = X_t dY_t + Y_t dX_t + d[X, Y]_t.$$

$$\tilde{S}_t = \frac{S_t}{B_t} = S_t \frac{1}{B_t} = S_t e^{-rt}.$$

We apply Itô's product formula with $X_t = S_t$ and $Y_t = e^{-rt}$.
First, note that e^{-rt} is a monotone, continuous, bounded function on $[0, T]$, so its quadratic variation is 0. Therefore, according to p. 29 of the Lecture notes, $[S, e^{-r\cdot}]_t = 0$. Secondly,

$$\frac{de^{-rt}}{dt} = -r \cdot e^{-rt} \implies de^{-rt} = -r \cdot e^{-rt} dt$$

Some Old Homework Exercises IV

Then,

$$\begin{aligned} d\tilde{S}_t &= e^{-rt}dS_t + S_t de^{-rt} + \underbrace{[S, e^{-r\cdot}]_t}_{=0} \\ &= e^{-rt}\mu S_t dt + e^{-rt}\sigma S_t dW_t - re^{-rt}S_t dt \\ &= e^{-rt}S_t\left((\mu - r)dt + \sigma dW_t\right) \\ &= \tilde{S}_t\left((\mu - r)dt + \sigma dW_t\right) \\ &= (\mu - r)\tilde{S}_t dt + \sigma \tilde{S}_t dW_t. \end{aligned}$$

What we received is again a geometric Brownian motion, but with a "discounted drift".

$$\tilde{S}_t = \tilde{S}_0 e^{\left(\mu - r - \frac{1}{2}\sigma^2\right)t + \sigma W_t}$$

To answer the last question, recall the dynamics of the discounted process.

$$d\tilde{S}_t = \underbrace{(\mu - r)\tilde{S}_t dt}_{\text{finite first variation}} + \underbrace{\sigma \tilde{S}_t dW_t}_{\text{local martingale part}}$$

$\tilde{S}_t$ is a local martingale if the finite variation part becomes zero, i.e. for $\mu = r$.
See also Lemma 6.6 in the lecture notes. Outlook CTF2 (& remember Maths2): If the model admits an equivalent martingale measure, under which the discounted price process is a martingale, there are no arbitrage opportunities. The market is complete if and only if the martingale measure is unique.