QFIN SS2023

Mathematical Finance

23.05.2023

Optimal Stopping problems

Previously we only considered European options, that is an option with payoff only at terminal time N.

Now we consider options that can be exercised at any time before (or at) maturity. Pricing these options can be tricky as the seller doesn't know when the buyer exercises the option. The idea is now to price the option assuming the buyer exercises optimally. Finding these optimal stopping times and a fair price for the American option is formulated as optimal stopping problems.

The problem: Consider some filtered probability space $(\Omega, \mathcal{F}, P)$, $\mathbb{F} = (\mathcal{F}_n)_{n=0,\dots,N}$ and a non-negative, adapted process $H = (H_n)_{n=0,\dots,N}$ (the payoff process) and let

$$\mathbb{T} := \{\tau : \Omega \to \{0,\dots,N\},\ \tau \text{ is a } \mathbb{F} \text{ stopping time}\}$$

The associated optimal stopping problem is to find $\tau^* \in \mathbb{T}$ with

$$\mathbb{E}(H_{\tau^*}) = \sup\{\mathbb{E}(H_\tau) : \tau \in \mathbb{T}\} =: U^*$$

Comments:

1) $H_n(\omega)$ gives the payoff , stopped' at $t=n$ (e.g. discounted payoff of an American option or investment opportunity

2) U^* called value of the stopping problem.

Examples:

1) American put option. $H_n = B_n^{-1}(K - S_n)^+$

2) European put option maturity $T \leq n$. $H_n = B_n^{-1}(K - S_n)^+ \mathbb{1}_{\{n=T\}}$

3) The case where H is a martingale. By the optimal sampling theorem $\mathbb{E}(H_\tau) = H_0$ for any $\tau \in \mathbb{T}$, so that any $\tau \in \mathbb{T}$ is optimal and $U^* = H_0$

Example: martingale betting stategy
Player places a bet on a simple game (e.g. toss coin). If he wins he gets 100 € other wise he loses it.
If he loses, he doubles the bet, if he wins he stops and makes a profit equal to the original stake.

The payoff is a martingale, therefore this stopping time is not superior and it is impossible to win on average (given finite lifetime of the player).

The Snell envelope

is an important tool in the analysis of optimal stopping problems.

Definition: The Snell envelope $(U_n)_{n=0,\dots,N}$ of the payoff process H is defined recursively by $U_N := H_N$ and

$$U_n := \max\{\underbrace{H_n}_{\text{payoff now}}, \underbrace{\mathbb{E}[U_{n+1} \mid \mathcal{F}_n]}_{\substack{\text{expected value}\\ \text{next time period}}}\} \quad \text{for} \quad n = N-1, N-2, \dots, 0$$

Proposition: Snell envelope for the payoff process H is the smallest supermartingale $\tilde{U}$ such that $\tilde{U}_n \geq H_n \quad \forall n \in \{0,\dots,N\}$

Characterization of stopping time

Theorem: Consider an optimal stopping problem with payoff process H and associated Snell envelope U. Then the following holds

① A stopping time $\tau \in \mathcal{T}$ is optimal if and only if the following two conditions hold
   i) $H_\tau = U_\tau$
   ii) The stopped process U^τ with $U^\tau_n := U_{\tau \wedge n}$ is a martingale

② An optimal stopping time is given by $\tau^* = \tau_{min} := \inf\{n = 0,\dots,N : U_n = H_n\}$

③ The Snell envelope gives the value of the problem: $U^* = U_0$

Proof Idea: Since U is a supermartingale with $H_\tau \leq U_\tau$ we have

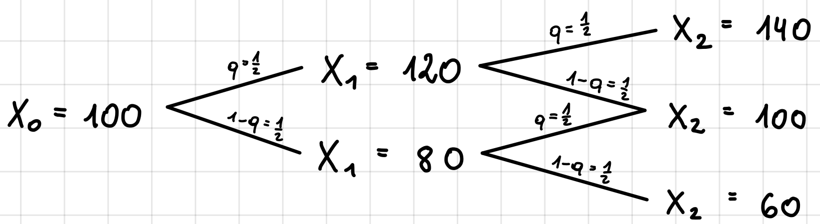
$$\mathbb{E}[H_\tau] \overset{i)}{\leq} \mathbb{E}[U_\tau] = \mathbb{E}[U^\tau_N] \overset{ii)}{\leq} U_0 \qquad (20)$$

If τ satisfies i) and ii) then we have equality and τ is optimal.

It is easy to see that τ_{min} satisfies ①: i) obvious, ii) for $n < \tau_{min}$ $H_n < U_n = \max\{H_n, \mathbb{E}[U_{n+1} \mid \mathcal{F}_n]\} = \underbrace{\mathbb{E}[U_{n+1} \mid \mathcal{F}_n]}_{\text{martingale}}$

It follows that $\tilde{\tau} = \tau_{min}$ and $U^* = U_0$ (because we have equality in (20) for τ_{min}).

Exercise: Consider a 2-period model that evolves $(r=0)$:



Riskneutral probabilities:

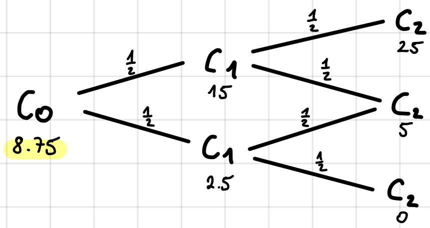
$$120 = q\,140 + (1-q)100 \iff q = \frac{120-100}{140-100} = \frac{1}{2}$$
$$80 = q\,100 + (1-q)60 \iff q = \frac{80-60}{40} = \frac{1}{2}$$
$$100 = q\,120 + (1-q)80 \iff q = \frac{100-80}{40} = \frac{1}{2}$$

$\Omega = \{(u,u),(u,d),(d,d),(u,u)\}$

$Q(\{(u,u)\}) = \frac{1}{2}\cdot\frac{1}{2} = \frac{1}{4}$

$Q(\{\omega\}) = \frac{1}{4} \quad \forall \omega \in \Omega$

European Call Spread: $H_n = ((X_2-95)^+ - (X_2-120)^+)\,\mathbb{1}_{\{n=2\}}$



$\frac{1}{2}\cdot 25 + \frac{1}{2}\cdot 5 = 15$

$\frac{1}{2}\cdot 5 + \frac{1}{2}\cdot 0 = 2.5$

$\frac{1}{2}\cdot 15 + \frac{1}{2}\cdot 2.5 = 8.75$

Replicating strategy:

$\theta_{2,0} + \theta_{2,1}\cdot 140 = 25$

$\theta_{2,0} + \theta_{2,1}\cdot 100 = 5$

$\theta^u_{2,1} = \frac{25-5}{140-100} = \frac{1}{2}$

$\theta^u_{2,0} = 5 - \frac{1}{2}100 = -45$

$\theta_{2,0} + \theta_{2,1}\cdot 100 = 5$

$\theta_{2,0} + \theta_{2,1}\cdot 60 = 0$

$\theta^d_{2,1} = \frac{5}{40} = \frac{1}{8}$

$\theta^d_{2,0} = -7.5$

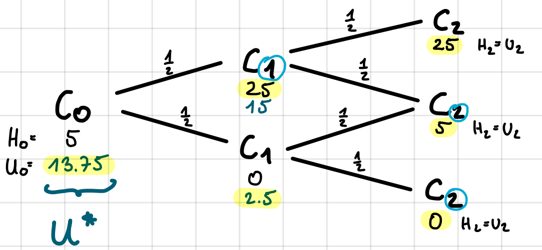
$\theta_{1,0} + \theta_{1,1}\cdot 120 = 15$

$\theta_{1,0} + \theta_{1,1}\cdot 80 = 2.5$

$\theta_{1,1} = \frac{12.5}{40} = \frac{5}{16}$

$\theta_{1,0} = 2.5 - \frac{5}{16}\cdot 80 = -22.5$

American Call Spread: $H_n = (X_n-95)^+ - (X_n-120)^+$



$25\cdot\frac{1}{2} + 5\cdot\frac{1}{2} = 15$

$5\cdot\frac{1}{2} + 0\cdot\frac{1}{2} = 2.5$

$25\cdot\frac{1}{2} + 2.5\cdot\frac{1}{2} = 13.75$

$$\tau_{min} = \begin{cases} 1 & \omega \in \{(u,u),(u,d)\} \\ 2 & \omega \in \{(d,u),(d,d)\} \end{cases}$$

Theorem: The fair price of an American claim with payoff process $(C_t)_{t=0,\dots,N}$ in an arbitrage-free and and complete security market $\mathcal{M}$ with martingale measure Q is given by $V_0^\Theta = S_{0,0}\,U_0^{\tilde{C}}$ where $(U_t^{\tilde{C}})_{t=0,1,\dots,N}$ is the Snell envelope of the discounted payoff process $(\tilde{C}_t) = \left(\frac{C_t}{S_{1,0}}\right)_{t=0,\dots,N}$.

Brownian motion

Definition: stoch process $X=(X_t)_{t\geq}$ on $(\Omega, \mathcal{F}, P)$ is a standard BM if

(i) $X_0 = 0$ a.s.

(ii) X has ind. increments: $X_{t+u} - X_t$ ind. of X_s $\forall s \leq t$

(iii) X has stationary incr.: $X_{t+u} - X_t \sim \mathcal{N}(0,u)$

(iv) continuous sample paths

Properties:

(i) $W_t \sim \mathcal{N}(0,t)$ $\quad (W_t = W_t - W_0 \sim \mathcal{N}(0,t-0))$

(ii) $\mathrm{Cov}(W_t, W_s) = \min(s,t)$

Proof: Let $s \leq t$. $\mathrm{Cov}(W_t,W_s) = \mathbb{E}[W_tW_s] - \underbrace{\mathbb{E}[W_s]}_{=0}\underbrace{\mathbb{E}[W_t]}_{=0} = \mathbb{E}[W_tW_s] = \mathbb{E}[(W_t - W_s + W_s)W_s]$

$$= \mathbb{E}[(W_t-W_s)W_s + W_s^2] = \mathbb{E}[\underbrace{(W_t-W_s)W_s}_{\text{indep. incr.}}] + \underbrace{\mathbb{E}[W_s^2]}_{\text{var}(W_s)=s} \quad (W_s \sim \mathcal{N}(0,s))$$

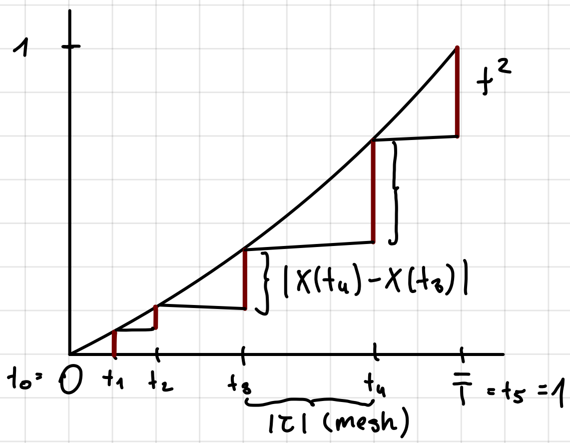
$$= \underbrace{\mathbb{E}[W_t-W_s]}_{=0}\underbrace{\mathbb{E}[W_s]}_{=0} + s = s$$

(iii) Finite dim. distr.: $t_1 < t_2 < \ldots < t_n$, $(W_{t_1}, W_{t_2}, \ldots, W_{t_n}) \sim \mathcal{N}(0,\Sigma)$ with $\Sigma = \begin{pmatrix} t_1 & t_1 & \cdots & t_1 \\ t_1 & t_2 & t_2 \cdots & t_2 \\ \vdots & t_2 & \ddots & \vdots \\ t_1 & t_2 & \cdots & t_n \end{pmatrix}$

(iv) $W = (W_t)_{t\geq 0}$ is a martingale

(v) $W = (W_t^2 - t)_{t \geq 0}$ is a martingale

First Variation:



Partition of interval $[0,T]$: $\tau = \{0 = t_1 < t_2 < \ldots < t_N = T\}$

The mesh is $|\tau| := \sup_{1\leq i \leq n} |t_i - t_{i-1}|$

First variation: function $X: [0,T] \to \mathbb{R}$.

$$\mathrm{Var}(X) := \sup_{\tau} \Big\{ \sum_{t_i \in \tau} |X(t_i) - X(t_{i-1})| \Big\}$$

Finite variation: $\mathrm{Var}(X) < \infty$

Example 1: $X(t) = t^2$ on $[0,1]$

$$\mathrm{Var}(X) = X(1) - X(0) = 1$$

Exercise: BM is a martingale: $\mathbb{E}[|B_t|] < \infty$, B_t adapted

$$\mathbb{E}[B_t \mid \mathcal{F}_s] = \mathbb{E}[B_t - B_s + B_s \mid \mathcal{F}_s] = \underbrace{\mathbb{E}[B_t - B_s \mid \mathcal{F}_s]}_{\Rightarrow \text{(ii)}} + \underbrace{\mathbb{E}[B_s \mid \mathcal{F}_s]}_{\text{measurable}} = \underbrace{\mathbb{E}[B_t - B_s]}_{=0 \ \text{(iii)}} + B_s = B_s$$

$$\mathbb{E}\left[e^{\sigma W_t - \frac{1}{2}\sigma^2 t} \mid \mathcal{F}_s\right] = \mathbb{E}\left[e^{\sigma(W_t - W_s)} e^{\sigma W_s} e^{-\frac{1}{2}\sigma^2 t} \mid \mathcal{F}_s\right] = e^{\sigma W_s} e^{-\frac{1}{2}\sigma^2 t} \, \mathbb{E}\left[e^{\overbrace{\sigma(W_t - W_s)}^{\mathcal{N}(0,\sigma^2(t-s))}}\right] = e^{\sigma W_s - \frac{1}{2}\sigma^2 t} e^{\frac{1}{2}\sigma^2 (t-s)}$$

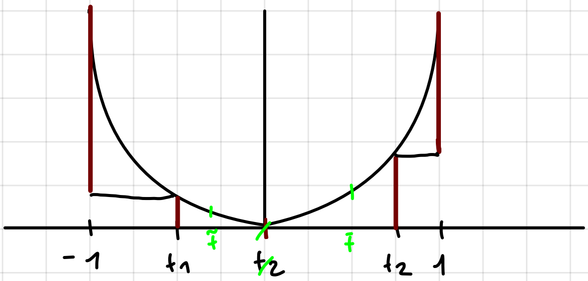
We can see that $\sum_{t_i\in\tau} |X(t_i)-X(t_{i-1})|$ is indep. of τ.

Proposition: For a right-cont, increasing function $X:[0,T]\to\mathbb{R}$, we have $\mathrm{Var}(X)=X(T)-X(0)$

Proof: Fix partition τ.

$$\sum_{t_i\in\tau}|X(t_i)-X(t_{i-1})| \overset{\text{incr. } X}{=} \sum_{t_i\in\tau} X(t_i)-X(t_{i-1}) = (\cancel{X(t_1)}-X(t_0))+(\cancel{X(t_2)}-\cancel{X(t_1)})+\ldots+(X(t_N)-\cancel{X(t_{N-1})}) = X(t_N)-X(t_0)$$

Example 2: $X(t)=t^2$ on $[-1,1]$



Choose partition τ:

$$\sum_{t_i\in\tau}|X(t_i)-X(t_{i-1})| = \sum_{t_i,t_{i-1}\geq 0}|X(t_i)-X(t_{i-1})| + \sum_{t_i,t_{i-1}\leq 0}|X(t_i)-X(t_{i-1})|$$
$$+\sum_{t_i\geq 0,\, t_{i-1}\leq 0}|X(t_i)-X(t_{i-1})|$$
$$= X(1)-X(\bar{t})+(X(-1)-X(\tilde{t}))+|X(\bar{t})-X(\tilde{t})|$$
$$= 2-\underbrace{\left((X(\bar{t})+X(\tilde{t})-|X(\bar{t})-X(\tilde{t})|\right)}_{\geq 0\ (=0 \text{ für } \tilde{t}=\bar{t}=0)}$$

$$|X(\bar{t})-X(\tilde{t})| \leq |X(\tilde{t})|+|X(\bar{t})| = X(\tilde{t})+X(\bar{t})$$

Remark: Every cont. diff. function has finite variation! (Proof is exercise)

Quadratic variation

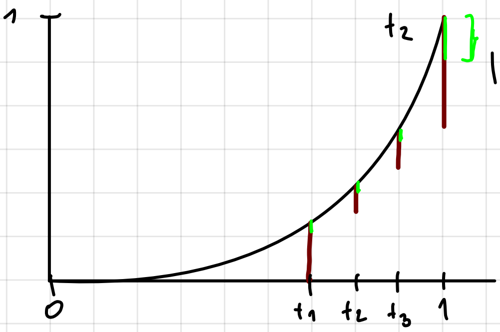
Fix sequence $|\tau_n|\to 0$ of partitions of $[0,T]$

$$V_t^2(X;\tau_n) := \sum_{\substack{t_i\in\tau \\ t_i\leq t}} (X(t_i)-X(t_{i-1}))^2$$

Define $[X]_t := \lim_{|\tau_n|\to 0} V_t^2(X;\tau_n)$. If it exists for $\forall t$, we say X admits **quadr. variation.**

Example

$X(t)=t^2$ on $[0,1]$



first variation $=1$ quadr. variation along τ_n

$|X(t_3)-X(t_4)|^2$

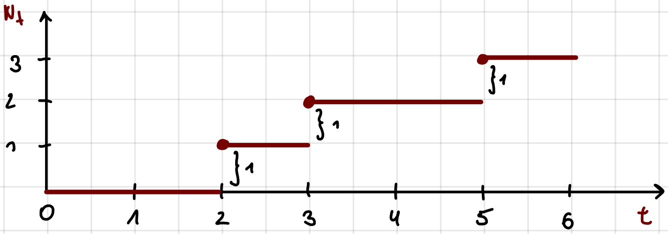
$\Rightarrow [X]_t = 0$

Theorem: If X is contin. and fin. variation $\Rightarrow [X]_t = 0$.

Proof: Fix τ_n. $V_t^2(X,\tau_n) = \sum_{\substack{t_i\in\tau_n\\ t_i\leq t}} (X(t_i)-X(t_{i-1}))^2 \leq \sup_{t_i\in\tau_n}(X(t_i)-X(t_{i-1})) \sum_{\substack{t_i\in\tau_n\\ t_i\leq t}} (X(t_i)-X(t_{i-1})) \leq \underbrace{\sup_{t_i\in\tau_n}(X(t_i)-X(t_{i-1}))}_{\xrightarrow{|\tau_n|\to 0}\, 0\ (X \text{ is cont.})} \mathrm{Var}(X)$

Corollary: X is cont. and $t \mapsto [X]_t$ strictly increasing $\Rightarrow$ X is of infin. variation on every subinterval $[0,T]$

Example: First and quad. of jump processes. Consider sample path of Poisson process:



$N_0 = 0$
$N_2 = 1$
$N_3 = 2$
$N_5 = 3$

Compute $Var(N)$ on $[0,6]$: $Var(N) = 3$

In general: $Var(N) = N_t$ on $[0,t]$
and $[N]_t = N_t$ as jumps $= 1$

Quad. variation for semimartingales

$$Y_t = \underbrace{M_t}_{\text{martingale}} + \underbrace{A_t}_{\text{"local trend"}} \quad \text{semimartingale decomposition}$$

A can be witten as $\int_0^t a_s \, ds$, this is a **finite variation process.**

Theorem: X cont. with quadr. variation $[X]_t$ and A cont. and finite variation. Define $Y = X + A$. Then

$$[Y]_t = [X + \underbrace{A}_{\text{fin. var.}}]_t = [X]_t$$

Quadratic Variation of BM

Theorem: Sequ. τ_n of $[0,T]$ with $|\tau_n| \to 0$. Then we have

$$\mathbb{E}\left[\left(V_t^2 (\underbrace{B_\cdot}_{\text{random (BM)}} ; \tau_n) - t \right)^2 \right] \to 0$$

Comments: (i) $\mathcal{L}^2$-convergence: $V_t^2 (B_\cdot ; \tau_n) \xrightarrow{\mathcal{L}^2} t$

(ii) $\Rightarrow$ Convergence in probability: $P(|V_t^2(B_\cdot ; \tau_n) - t| \geq \varepsilon) \to 0$

(iii) $\Rightarrow$ $\exists$ subsequ. of τ_n with $V_t^2(B_\cdot(\omega), \tau_n) \xrightarrow{n \to \infty} t$ a.s.

therefore $[B(\omega)]_t = t$ for (almost) all $\omega \in \Omega$ (pathwise)

(iv) Sample paths of BM are infinite (Theorem)

Proof: $\mathbb{E}\left[\left(\sum_{t_i \in \tau} (B_{t_i} - B_{t_{i-1}})^2 - (t_i - t_{i-1}) \right)^2 \right] = \mathbb{E}\left[\sum_{t_j, t_i \in \tau} \left((B_{t_i} - B_{t_{i-1}})^2 - (t_i - t_{i-1}) \right) \left((B_{t_j} - B_{t_{j-1}})^2 - (t_j - t_{j-1}) \right) \right]$

For $i \neq j$: increm. are indep. and $\mathbb{E}[(B_{t_i} - B_{t_{i-1}})^2 - (t_i - t_{i-1})] = 0$

$$\mathbb{E}\left[\sum_{t_i \in \tau} \left((B_{t_i} - B_{t_{i-1}})^2 - (t_i - t_{i-1}) \right)^2 \right] \overset{Var(X^2) = 2\sigma^4,\ X \sim \mathcal{N}(0,\sigma^2)}{=} \sum_{t_i \in \tau} 2 (t_i - t_{i-1})^2 \leq 2 \sum_{t_i \in \tau} (t_i - t_{i-1}) \overbrace{\sup |t_i - t_{i-1}|}^{|\tau_n|} = 2t|\tau_n| \xrightarrow{|\tau_n| \to 0} 0$$

Chapter: Ito Calculus

Introduction:

Discrete time model with trading dates $t_1 < t_2 < \ldots < t_N$, and θ some trading strategy

Gains from trade: $G^\theta_{t_N} = \sum_{i=1}^{N} \theta_{t_i} (\underset{\uparrow \text{ Stock at } t_i}{S_i} - \underset{\nwarrow \text{ Stock at } t_{i-1}}{S_{t_{i-1}}})$

In continuous time we define for a trading strategy $(\theta)_{t \le T}$

Gains from trade: $$\lim_{\substack{N\to\infty \\ |t_n - t_{n-1}| \to 0}} G^\theta_{t_N} = \lim_{\substack{N\to\infty \\ |t_n - t_{n-1}| \to 0}} \sum_{i=1}^{N} \theta_{t_i} (S_i - S_{t_{i-1}}) =: \int_0^{t_N} \theta_t \, dS_t$$

Problem: Existence?

If S has infinite variation (e.g. $S_t = B_t$, $\sup_\tau \sum_{t_i} |B_{t_i} - B_{t_{i-1}}| = \infty$) then the limit does not exist for all continuous θ.

Maybe the limit exists for a smaller set of θ (not necessarily all continuous θ but a subset)?

$\Rightarrow$ use martingale theory to proof existence with so called Ito formula. $\longrightarrow$ we need it for the Black-Scholes model (as it is built with BM).

Motivation: Ito formula

"Stochastic counterpart of the chain rule"

① Chain rule: C^1-functions $f: \mathbb{R} \to \mathbb{R}$ and $X: \mathbb{R}^+ \to \mathbb{R}$ with derivatives $f'(x)$ and $X'(t)$

$$f(X(t)) \underset{\substack{\uparrow \\ \text{fund. th. of} \\ \text{calculus}}}{=} f(X(0)) + \underbrace{\int_0^t \underbrace{f'(X(s))X'(s)}_{\text{Chain rule}} \, ds}_{=: \int_0^t f'(X(s)) \, dX_s} \qquad (\text{"ordinary calculus"})$$

② Now, X is not C^1 but continuous and finite variation.

$$f(X_t) = f(X_s) + \underbrace{\lim_{n\to\infty} \sum_{t_i \in \tau_n} f'(X_{t_{i-1}})(X_{t_i} - X_{t_{i-1}})}_{=: \int_0^t f'(X_s) \, dX_s \quad (\text{exists as } X \text{ has f.V})}$$

③ Now X is continuous and infinite variation but quadr. variation

$$f(X_t) = f(X_s) + \int_0^t f'(X_s) \, dX_s + \text{correction term} \qquad (\text{for } f \in C^2)$$

Theorem (Itô-formula): Consider cont. $X: [0,T] \to \mathbb{R}$ with contin. quad. variation $[X]_t$ and a C^2-function $F: \mathbb{R} \to \mathbb{R}$.

Then we have for $t \le T$:

$$F(X_t) = F(X_0) + \underbrace{\int_0^t F'(X_s) \, dX_s}_{:= \lim_{n\to\infty} \sum_{\substack{t_i \in \tau_n \\ t_i \le t}} F'(X_{t_{i-1}})(X_{t_i} - X_{t_{i-1}}) \; (*)} + \frac{1}{2} \int_0^t F''(X_s) \, d[X]_s$$

$(*)$ is called Itô-Integral.

Comments:

1. For X finite variation ($\Rightarrow [X]_t = 0$) the formula is just $f(X_t) = f(X_0) + \int_0^t f'(X_s)\, dX_s$

2. The sums defining $\int_0^t F'(s)\, dX_s$ are non-anticipating ($F'(t_{i-1})$ used)

3. Short-hand notation: $dF(X_t) = F'(X_t)\, dX_t + \frac{1}{2} F''(X_t)\, d[X]_t$

Examples: Ito formula

1. Compute B_t^2: $F(B_t) = B_t^2$ (here $X_t = B_t$)

$F(X_t) = F(B_t) = B_t^2 = \dots \Rightarrow$ Ito formula

$F(x) = x^2$, $F'(x) = 2x$, $F''(x) = 2$

$$F(B_t) = F(B_0) + \int_0^t F'(B_s)\, dB_s + \frac{1}{2}\int_0^t F''(B_s)\, d\underbrace{[B]_s}_{s}$$

$$= B_0^2 + \int_0^t 2 B_s\, dB_s + \underbrace{\frac{1}{2}\int_0^t 2\, ds}_{=t}$$

$$= 2\int_0^t B_s\, dB_s + t$$

$$\Rightarrow \int_0^t B_s\, dB_s = \frac{1}{2}(B_t^2 - t) = \frac{B_t^2}{2} - \underbrace{\frac{t}{2}}_{\text{correction term}}$$

Compare with 'ordinary' calculus: Compute $\int_0^t f(s)\, df(s)$ for C^1-function f.

$$\int_0^t f(s)\, df(s) = \int_0^t f(s) f'(s)\, ds = \begin{bmatrix} u = f(s) \\ du = f'(s)ds \end{bmatrix} = \int_{f(0)=0}^{f(t)} u\, du = \frac{f(t)^2}{2}$$

2. Compute e^{X_t}: $F(x) = e^x$, $F'(x) = e^x$, $F''(x) = e^x$

$$F(X_t) = e^{X_t} = e^{X_0} + \int_0^t e^{X_s}\, dX_s + \frac{1}{2}\int_0^t e^{X_s}\, d[X]_s$$

If $X_t = B_t$: $e^{B_t} = 1 + \int_0^t e^{B_s}\, dB_s + \frac{1}{2}\int_0^t e^{B_s}\, ds$

Pictures:

1) $\int_0^t B_s\, ds$

2) $\int_0^t B_s\, dB_s = \frac{1}{2}(B_t^2 - t)$

Quadratic Variation of the Ito-Integral

Lemma: Consider a cont. function $X(t)$ with cont. quad. variation $[X]_t$ and $F \in C^1(\mathbb{R})$. Then $t \mapsto F(X_t)$ has quad. variation $[F(X)]_t = \int_0^t (F'(X_s))^2 [X]_s$

Theorem: For $f \in C^1(\mathbb{R})$ the Ito-integral $I_t = \int_0^t f(X_s)\, dX_s$ is well defined; its quad. variation equvals

$$[I]_t = \int_0^t f^2(X_s)\, d[X_s]$$

<u>Proof:</u> Apply lemma to $F = \int f(x)\,dx$. It holds $F' = f$ and $F'' = f'$.

1) Well-defined: $f \in C^1 \Rightarrow F \in C^2 \Rightarrow$ Ito formula proofs existence

2) $F(X_t) = F(X_0) + \underbrace{\int_0^t \underbrace{F'(X_s)}_{f}\, dX_s}_{=: I_t} + \underbrace{\frac{1}{2}\int_0^t \underbrace{F''(X_s)}_{f'}\, d[X]_s}_{=: A_t} = F(X_0) + \underbrace{I_t}_{\text{Ito Integral}} + \underbrace{A_t}_{\text{finite variation (FV)}}$

$$\Rightarrow \underset{\int_0^t \underbrace{F'(X_s)}_{f} d[X]_s}{\overset{\|}{[F(X)]_t}} = [\underbrace{F(X_0)}_{\text{constant} \Rightarrow FV} + I + \underbrace{A}_{FV}]_t = [I]_t$$

Examples:

1) Compute quad. var. $[X]_t$ for $X_t = B_t^2$

$$B_t^2 \overset{\text{Ito-formula}}{=} B_0^2 + \int_0^t 2B_s\, dB_s + \frac{1}{2}\int_0^t 2\, d\underbrace{[B]_s}_{s}$$

$$[X]_t = [B^2]_t = [\int_0^\cdot 2B_s\, dB_s]_t = \int_0^t 4B_s^2\, d\underbrace{[B]_s}_{s} = 4\int_0^t B_s^2\, ds$$

2) Compute quad. var. $[X]_t$ for $X_t = \exp(B_t)$

$$\exp(B_t) = \exp(B_0) + \int_0^t e^{B_s}\, dB_s + \frac{1}{2}\int_0^t e^{B_s}\, ds$$

$$[\exp(B)]_t = \int_0^t e^{2B_s}\, ds$$

Proof of the Ito-Formula

As a first step we establisch the following

Lemma: For every piecewise constant function $g : [0,T] \to \mathbb{R}$ we have

$$\lim_{n\to\infty} \sum_{\substack{t_i \in \tau_n \\ t_i \le t}} g(t_{i-1}) (X_{t_i} - X_{t_{i-1}})^2 = \int_0^t g(s)\, d[X]_s \qquad (*)$$

<u>Proof:</u> Recall $V_t^2(X; \tau_n) = \lim_{n\to\infty} \sum_{t_i \le t} (X_{t_i} - X_{t_{i-1}})^2$.

For $g(t) = \mathbb{1}_{(a,b]}(t)$ $(*)$ translates to $\lim_{n\to\infty} \sum_{\substack{t_i \in \tau_n \\ t_i \le t}} \mathbb{1}_{(a,b]}(t_i)(X_{t_i} - X_{t_{i-1}})^2 = \lim_{n\to\infty} \sum_{\substack{t_i \in \tau_n \\ a < t_i \le b}} (X_{t_i} - X_{t_{i-1}})^2 =$

$$= \lim_{n\to\infty} V_b^2(X, \tau_n) - V_a^2(X, \tau_n) = [X]_b - [X]_a$$

exists as X allows quad. var.

For a general piecewise const. function g we approximate g by step functions.

Now the theorem itself.

Consider $t_i, t_{i-1} \in \tau_n$ such that $t_i \le t$ and $(\Delta X)_{i,n} = (X_{t_i} - X_{t_{i-1}})$. We get from Taylor expansion:

$$F(X_{t_i}) - F(X_{t_{i-1}}) = F'(X_{t_{i-1}})(\Delta X_{i,n}) + \tfrac{1}{2} F''(X_{\tilde{t}})(\Delta X_{i,n})^2 \qquad , \tilde{t} \in [t_{i-1}, t_i]$$

$$= F'(X_{t_{i-1}})(\Delta X_{i,n}) + \tfrac{1}{2} F''(X_{t_{i-1}})(\Delta X_{i,n})^2 + R_{i,n} \qquad , R_{i,n} = \tfrac{1}{2}\left(F''(X_{\tilde{t}}) - F''(X_{t_{i-1}})\right)(\Delta X_{i,n})^2$$

Define $\delta_n = \max\{ |X_t - X_{i-1}|, t \in [t_{i-1}, t_i], t_i \in \tau_n \} \xrightarrow{n\to\infty} 0$ as X cont. and $|\tau_n| \to 0$.

Moreover,

$$|R_{i,n}| \le \underbrace{\left(\tfrac{1}{2} \max_{|x-y| < \delta_n} |F''(x) - F''(y)|\right)}_{=: \varepsilon_n} (\Delta X_{i,n})^2$$

$\varepsilon_n \xrightarrow{n\to\infty} 0$ (F'' is continuous) as $\delta_n \xrightarrow{n\to\infty} 0$

Hence,

$$\Big|\sum_{t_i \in \tau_n} R_{i,n}\Big| \le \sum_{t_i \in \tau_n} |R_{i,n}| \le \varepsilon_n \underbrace{\sum_{t_i \in \tau_n} (\Delta X_{i,n})^2}_{< \infty \ ([X]_t \text{ exists})} \xrightarrow{n\to\infty} 0$$

Now,

$$F(X_t) - F(X_0) = \lim_{n\to\infty} \sum_{\substack{t_i \in \tau_n \\ t_i \le t}} F(X_{t_i}) - F(X_{t_{i-1}}) = \underbrace{\lim_{n\to\infty} \sum_{\substack{t_i \in \tau_n \\ t_i \le t}} F'(X_{t_{i-1}})(\Delta X_{i,n})}_{=: \int_0^t F'(X_s)\, dX_s} + \underbrace{\lim_{n\to\infty} \sum_{\substack{t_i \in \tau_n \\ t_i \le t}} \tfrac{1}{2} F''(X_{t_{i-1}})(\Delta X_{i,n})^2}_{= \frac{1}{2}\int_0^t F''(X_s)\, d[X]_s} + \underbrace{\lim_{n\to\infty} \sum_{\substack{t_i \in \tau_n \\ t_i \le t}} R_{i,n}}_{= 0}$$

We obtain the Itô-formula and show that $\int_0^t F'(X_s)\, dX_s$ exists.

Martingale-property of the Ito-Integral

Process $I_t(\omega) = \int_0^t f(X_s(\omega))\, dX_s(\omega)$ is a stoch. process (as BM)

Consider $f \in C^1$ and martingale M: $I_t := \int_0^t f(M_s)\, dM_s$; is I_t also a martingale? **Yes!** (but a „weaker" definition)

Problems with integrability can arise: Therefore only a weaker result is true.

Definition: A stoch. process M is called **local martingale** if there are stopping times $T_1 \le T_2 \le \dots$ such that

(i) $\lim_{n\to\infty} T_n(\omega) = \infty$ a.s.

(ii) $(M_{T_n \wedge t})_{t \ge 0}$ is a martingale for all n.

True martingale $\overset{\not\Leftarrow}{\Rightarrow}$ local martingale

Last lecture:

Ito-formula: $F(X_t) = F(X_0) + \underbrace{\int_0^t F'(X_s)\, dX_s}_{=: M_t} + \underbrace{\frac{1}{2}\int_0^t F''(X_s)\, d[X]_s}_{=: A_t}$

X_t is local martingale: **Semimartingale decomposition**
e.g. $X_t = B_t$

$$\underbrace{F(X_t)}_{\text{semimartingale}} = \underbrace{M_t}_{\text{local martingale}} + \underbrace{A_t}_{\text{FV process}}$$

Quad. variation: $[F(X)]_t = [M]_t = \int_0^t (F'(X_s))^2\, d[X]_s$

Example: Semimartingale decomposition for B_t^2?

$$B_t^2 \overset{\text{Ito}}{=} \underbrace{\int_0^t 2B_s\, dB_s}_{=: M_t} + \underbrace{\int_0^t 1\, ds}_{=: A_t}$$

BM is martingale

$$[B]_t^2 = \int_0^t 4B_s^2\, ds$$

Example: local martingale that is not a true martingale

$$I_t = \int_0^t \Theta_s \, dW_s \qquad \text{where } \Theta_s(\omega) = Y(\omega) \sim \text{r.v. with } \mathbb{E}[|Y|] = \infty \text{ (e.g. Cauchy distr.) ind. of } W$$
$$= Y \cdot \int_0^t dW_s = Y W_t$$

Integrability: $\mathbb{E}[|YW_t|] = \underbrace{\mathbb{E}[|Y|]}_{=\infty} \underbrace{\mathbb{E}[|W_t|]}_{<\infty} = \infty$ ↯

Consider $T_n = \inf\{t : |I_t| \geq n\} \to \infty$ (localizing sequence)

$$\mathbb{E}[|M_{T_n \wedge t}|] = \mathbb{E}[|YW_t| \mathbb{1}_{\{|YW_t| < n\}}] + \mathbb{E}[|M_{T_n}| \mathbb{1}_{\{M_{T_n} = n\}}] \leq n \qquad \forall n$$

Martingale prop: $\mathbb{E}[M_{t \wedge T_n} | \underbrace{\mathcal{F}_s}_{\sigma(\{Y, W_r, r \leq s\})}] = M_{s \wedge T_n}$

$$\mathbb{E}[M_{t\wedge T_n} | \mathcal{F}_s] = \mathbb{E}[Y W_{t \wedge T_n} | \mathcal{F}_s] = Y \, \mathbb{E}[\underbrace{W_{t \wedge T_n}}_{\substack{\text{stopped martingale} \\ \text{is martingale}}} | \mathcal{F}_s] = Y W_{s \wedge T_n} = M_{s \wedge T_n} \checkmark$$

Adapted: $\mathcal{F}_s = \sigma(\{Y, W_r ; r \leq s\})$

Theorem: Continuous local martingale M with continuous quad. variation $[M]_t$ and $f \in \mathcal{C}^1$. Then

$$I_t(\omega) = \int_0^t f(M_s(\omega)) \, dM_s(\omega) \quad \text{is a } \underline{\text{local}} \text{ martingale}$$

Proof: Assume M bounded martingale, f bounded (General case: localize with T_n). $|\tau_n| \to 0$, and fix n.

$$I_k^n := \sum_{\substack{t_i \in \tau_n \\ i \leq k}} f(M_{t_{i-1}})(M_{t_i} - M_{t_{i-1}}), \quad k \leq n \qquad \text{is martingale wrt } \{\mathcal{F}_k^n\}_k \quad \mathcal{F}_k^n := \mathcal{F}_{t_k} \text{ (discrete)}$$

as $$\mathbb{E}[I_k^n - I_{k-1}^n | \mathcal{F}_{k-1}^n] = \mathbb{E}[f(M_{t_{k-1}})(M_{t_k} - M_{t_{k-1}}) | \mathcal{F}_{k-1}^n] = f(M_{k-1}) \underbrace{\mathbb{E}[M_{t_k} - M_{t_{k-1}} | \mathcal{F}_{k-1}^n]}_{= 0 \text{ (as } M \text{ martingale)}}$$

For cont.: $t_n \searrow t$ and $s_n \searrow s$, $\mathbb{E}[I_t \mathbb{1}_A] = \mathbb{E}[I_s \mathbb{1}_A] \quad \forall A \in \mathcal{F}_s$ $(\Rightarrow \mathbb{E}[I_t | \mathcal{F}_s] = I_s)$

Proposition: Let M be a local martingale. Then the following is equivalent:

(i) M is a true martingale and $\mathbb{E}(M_t^2) < \infty$, $\forall t \geq 0$.

(ii) $\mathbb{E}[[M]_t] < \infty$

If either (i) or (ii) holds we have $\mathbb{E}[M_t^2] = \mathbb{E}[[M]_t]$

Examples:

1) $I_t := \int_0^t B_s \, d\underbrace{B_s}_{\text{martingale}}$ is a local martingale. Is $\int_0^t B_s \, dB_s$ a true martingale?

$$[I]_t = \int_0^t B_s^2 \, ds$$

Fubini Tonelli $\int\int |f| \, dx\, dy$ or $\int\int |f| \, dy \, dx$ exist

$$\mathbb{E}[[I]_t] = \mathbb{E}\left[\int_0^t B_s^2 \, ds\right] = \int_0^t \mathbb{E}[B_s^2] \, ds = \int_0^t s \, ds = \frac{t^2}{2} < \infty$$

2) Is $\underbrace{\int_0^t e^{B_s} \, dB_s}_{I_t :=}$ a true martingale?

$$\mathbb{E}[[I]_t] = \mathbb{E}\left[\int_0^t e^{2B_s} \, ds\right] = \int_0^t \mathbb{E}[\underbrace{e^{\overbrace{2B_s}^{\sim \mathcal{N}(0,4s)}}}_{\sim \log NV(0,4s)}] \, ds = \int_0^t e^{2s} \, ds = \frac{1}{2} e^{2s} \Big|_0^t = \frac{1}{2}(e^{2t} - 1) < \infty$$

3) Not a true martingale: $I_t := \int_0^t \frac{1}{|B_s|} \, dB_s$, $\mathbb{E}\left[\frac{1}{|B_s|}\right] = \infty$

> Why price process with infinite variation?
> Arbitrag-free: S is martingale
> S FV $\overset{\text{Theorem}}{\Longrightarrow}$ S is constant a.s.

Proposition: Consider a local martingale M with continuous trajectories of finite variation.

Then the paths of M are constant, ie. $M_t = M_0$ a.s.

<u>Proof:</u> If $[M]_t = 0$, then M^2 also loc. martingale: $M_t^2 = M_0 + \int 2M_s \, dM_s + \underbrace{\int \cdot \, d[M]_s}_{=0}$; Assume for simpl. M, M^2 true marting.

$$0 \leq \mathbb{E}[(M_t - M_0)^2] = \mathbb{E}[M_t^2 - 2M_t M_0 + M_0^2] = \underbrace{\mathbb{E}[M_t^2]}_{= M_0^2} - 2\underbrace{\mathbb{E}[M_t]}_{= M_0}\underbrace{\mathbb{E}[M_0]}_{= M_0} + \underbrace{\mathbb{E}[M_0^2]}_{= M_0^2} = 0$$

$$\Rightarrow \mathbb{E}[\underbrace{(M_t - M_0)^2}_{\geq 0}] = 0 \quad \Leftrightarrow \quad M_t = M_0 \text{ a.s.}$$

Covariation:

Fix τ_n partition of $[0,T]$, $\tau_n \to 0$, and continuous quad. variation $[X]_t, [Y]_t$

Definition: $\lim\limits_{n\to\infty} \sum\limits_{\substack{t_i \leq t \\ t_i \in \tau_n}} (X_{t_i} - X_{t_{i-1}})(Y_{t_i} - Y_{t_{i-1}}) =: [X,Y]_t$ **Covariation**

Theorem: $[X,Y]_t$ exists $\Leftrightarrow$ $[X+Y]_t$ exists

Polarization Identity: $[X,Y]_t = \frac{1}{2}\left([X+Y]_t - [X]_t - [Y]_t\right)$

Examples:

1) $[X,A]_t = 0$ if A FV-process, X with $[X]_t$, continuous

We know: $[X+\underbrace{A}_{\text{FV process}}]_t = [X]_t$, $[A]_t = 0$

$$\Rightarrow [X,A]_t = \frac{1}{2}\Big(\underbrace{[X+A]_t}_{[X]_t} - [X]_t - \underbrace{[A]_t}_{0}\Big) = 0$$

2) Indep. BM B^1 and B^2: $[B^1, B^2] = 0$

$$\Rightarrow [B^1,B^2]_t = \frac{1}{2}\Big(\underbrace{[B^1+B^2]_t}_{=2t} - \underbrace{[B^1]_t}_{=t} - \underbrace{[B^2]_t}_{=t}\Big) = 0$$

$X_t := \frac{B^1+B^2}{\sqrt{2}}$ is BM (Proof as an exercise)

$[\sqrt{2}X]_t = 2[X]_t = 2t$ (It holds $[aX]_t = a^2[X]_t$, and $[aX, bY]_t = ab[X,Y]_t$)

3) $f,g \in \mathcal{C}^1$, X cont and cont $[X]_s$

$$Y_t := \int_0^t f(X_s)\,dX_s, \quad Z := \int_0^t g(X_s)\,dX_s$$

Then: $[Y,Z]_t = \int_0^t f(X_s)g(X_s)\,d[X]_s$

Theorem: Given cont. functions $X = (X^1,\dots,X^d): [0,T] \to \mathbb{R}$ with

$$[X^k, X^l]_t = \begin{cases} [X^k]_t & \text{if } k=l \\ \frac{1}{2}([X^k,X^l]_t - [X^k]_t - [X^l]_t) & \text{if } k \neq j \end{cases}$$

and $F \in \mathcal{C}^2$, $F: \mathbb{R}^d \to \mathbb{R}$. Then

$$F(X_t) = F(X_0) + \sum_{i=1}^d \int_0^t \underbrace{\frac{\partial}{\partial x_i} F(X_s)}_{\substack{\text{partial derivatives} \\ \text{sometimes denoted } F_{x_i}}}\, dX_s^i + \frac{1}{2}\sum_{i,j=1}^d \int_0^t \underbrace{\frac{\partial^2}{\partial x_i \partial x_j} F(X_s)}_{\substack{\text{2nd order partial deriv.} \\ \text{somet. denoted } F_{x_i x_j}}}\, d[X^i, X^j]_s.$$

Short-Notation: $dF(X_t) = \sum_{i=1}^d F_{x_i}(X_t)\,dX_t^i + \frac{1}{2}\sum_{i,j=1}^d F_{x_i x_j}(X_t)\,d[X^i,X^j]_t$

Example. $W = (W^1,\dots,W^d)$ is a d-dim BM.

Note $[W^i, W^j]_t = \begin{cases} 0 & i\neq j \\ t & i=j \end{cases}$ (it holds $[W^i,W^i] = [W^i]_t$)

$$F(W_t) = \sum_{i=1}^d \int_0^t F_{x_i}(W_s)\,dW_s^i + \frac{1}{2}\sum_{i=1}^d \int_0^t F_{x_i x_i}(W_s)\,ds$$

Consider: $F(x_1,x_2,x_3) = x_1^2 + x_2^2 + x_3^2$ and $B = (B^1, B^2, B^3)$ 3-dim BM

$F_{x_i} = 2x_i$, $F_{x_i x_j} = 0$, $F_{x_i x_i} = 2$

$$F(B_t) = \int_0^t 2B_s^1\,dB_s^1 + \int_0^t 2B_s^2\,dB_s^2 + \int_0^t 2B_s^3\,dB_s^3 + \overbrace{\frac{1}{2}\sum_{i=1}^3 \int_0^t 2\,ds}^{=3t}$$

Corollary: (Itô's Product formula)

Given cont. $X, Y, [X]_t, [Y]_t$ and $[X,Y]_t$. Then

$$X_t Y_t = X_0 Y_0 + \int_0^t X_s \, dY_s + \int_0^t Y_s \, dX_s + [X,Y]_t$$

Proof: Apply Ito-formula: $F(x,y) = xy$, $F_x = y$, $F_y = x$, $F_{xx} = F_{yy} = 0$, $F_{xy} = 1$

$$X_t Y_t = X_0 Y_0 + \int_0^t X_s \, dY_s + \int_0^t Y_s \, dX_s + \underbrace{\frac{1}{2}\Big(\underbrace{\int_0^t 1 \, d[X,Y]_s}_{= [X,Y]_t - \underbrace{[X,Y]_0}_{=0}} + \int_0^t 1 \, d\underbrace{[Y,X]_s}_{=[X,Y]_s} \Big)}_{= [X,Y]_t}$$

(with x_1, x_2 above $[X,Y]_s$ and x_2, x_1 above $[Y,X]_s$)

Corllary: (Itô for time dependence)

Cont. X with cont. $[X]_t$ and $F(t,x)$ which is C^1 in t and C^2 in x. Then

$$F(t, X_t) = F(0, X_0) + \int_0^t F_t(t, X_s) \, ds + \int_0^t F_x(s, X_s) \, dX_s + \frac{1}{2} \int_0^t F''(s, X_s) \, d[X]_s$$

Example: Geometric Brownian Motion

BM W, constants $\alpha, \sigma > 0$ and initial value S_0, then the geom. BM is

$$S_t = S_0 \exp\left(\sigma W_t + \left(\mu - \frac{\sigma^2}{2}\right)t\right)$$

Apply Ito formula:

$$\begin{aligned}
F(t,x) &= S_0 \exp\left(\sigma x + \left(\mu - \tfrac{\sigma^2}{2}\right)t\right) &&\rightsquigarrow S_t \\
F_x(t,x) &= S_0 \sigma \exp\left(\sigma x + \left(\mu - \tfrac{\sigma^2}{2}\right)t\right) &&\rightsquigarrow \sigma S_t \\
F_t(t,x) &= S_0 \left(\mu - \tfrac{\sigma^2}{2}\right) \exp\left(\sigma x + \left(\mu - \tfrac{\sigma^2}{2}\right)t\right) &&\rightsquigarrow \left(\mu - \tfrac{\sigma^2}{2}\right) S_t \\
F_{xx}(t,x) &= S_0 \sigma^2 \exp\left(\sigma x + \left(\mu - \tfrac{\sigma^2}{2}\right)t\right) &&\rightsquigarrow \sigma^2 S_t
\end{aligned}$$

$$S_t = S_0 + \int_0^t \left(\mu - \cancel{\tfrac{\sigma^2}{2}}\right) S_s \, ds + \int_0^t \sigma S_s \, dW_s + \cancel{\frac{1}{2} \int_0^t \sigma^2 S_s \, ds}$$

$$= S_0 + \int_0^t \mu S_s \, ds + \int_0^t \sigma S_s \, dW_s$$

Last Lecture(s): Itô formula

1) $F: \mathbb{R} \to \mathbb{R}$, $F \in \mathcal{C}^2(\mathbb{R})$: $\quad F(X_t) = F(X_0) + \int_0^t F'(X_s)\,dX_s + \frac{1}{2}\int_0^t F''(X_s)\,d[X]_s$

2) $F: \mathbb{R}^d \to \mathbb{R}$, $F \in \mathcal{C}^2(\mathbb{R}^d)$: $\quad F(X_t) = F(X_0) + \sum_{i=1}^d \int_0^t F_{x_i}(X_s)\,dX_s^i + \sum_{i,j=1}^d \frac{1}{2}\int_0^t F_{x_i x_j}(X_s)\,d[X^i, X^j]_s$

3) $F: \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$, $F \in \mathcal{C}^{1,2}(\mathbb{R}^+ \times \mathbb{R})$: $\quad F(t, X_t) = F(0, X_0) + \int_0^t F_t(X_s)\,ds + \int_0^t F_x(X_s)\,dX_s + \frac{1}{2}\int_0^t F_{xx}(X_s)\,d[X]_s$

$F: \mathbb{R}^+ \times \mathbb{R}^d \to \mathbb{R}$
Combine 2) + 3)

Ito process:

... is the solution of a SDE driven by BM.

Definition: Given a BM $(W)_{t\geq 0}$, time point $t_0 > 0$, some $x \in \mathbb{R}$ and funtions $\mu: \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$, $\sigma: \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ the process

X that satisfies

$$X_t = \underbrace{x + \int_{t_0}^t \underbrace{\mu(s, X_s)}_{\text{"drift"}}\,ds}_{\text{FV}} + \underbrace{\int_{t_0}^t \underbrace{\sigma(s, X_s)}_{\text{"dispersion"}}\,dW_s}_{\text{local martingale}}$$

is called an Ito-process.

Short: $dX_t = \mu(t, X_t)\,dt + \sigma(t, X_t)\,dW_t$

Examples: Assume $t_0 = 0$

1) **Arithmetic BM:** $\quad \mu(t,x) = \alpha$, $\quad \sigma(t,x) = \beta$

$$\Rightarrow X_t = x + \int_0^t \alpha\,ds + \int_0^t \beta\,dW_s = x + \alpha t + \beta(W_t - W_0) = x + \alpha t + \beta \underbrace{W_t}_{\mathcal{N}(0,t)} \sim \mathcal{N}(x + \alpha t, \beta^2 t)$$

2) **Geometric BM:** $\quad \mu(t,x) = \alpha x$, $\quad \sigma(t,x) = \beta x \quad$ for $x > 0$.

$$\Rightarrow X_t = x \exp\left((\alpha - \tfrac{1}{2}\beta^2)t + \beta W_t\right)$$

3) **Ornstein Uhlenbeck process:**

$$dX_t = \underbrace{\kappa(\theta - X_t)}_{\mu(t,x) = \kappa(\theta - x)}\,dt + \underbrace{\beta}_{\sigma(t,x) = \beta}\,dW_t, \quad X_0 = \bar{r} \quad \text{for } \beta > 0$$

Solution (Exercise):

$$X_t = e^{-\kappa t}\bar{r} + \theta(1 - e^{-\kappa t}) + \beta e^{-\kappa t}\int_0^t e^{\kappa s}\,dW_s$$

Trick: Rewrite $\quad e^{\kappa t} X_t = \bar{r} + \theta(e^{\kappa t} - 1) + \beta \int_0^t e^{\kappa s}\,dW_s$

Properties: $dX_t = \mu(t,X_t)\,dt + \sigma(t,X_t)\,dW_t$

- $[X]_t = \int_{t_0}^{t} \sigma^2(s,X_s)\,ds$
- Markov Process: $E[f(X_T)|\mathcal{F}_t] = E[f(X_T)|X_t]$

  If σ and μ do not depend on time $(\mu(x), \sigma(x))$: $E[f(X_T)|\mathcal{F}_t](\omega) = E_{X_t(\omega)}[f(X_{T-t})]$

  where E_x is expectation wrt to distribution of X with $X_0 = x$.
- Semimartingales: $M_t = \int_0^t \sigma(s,X_s)\,dW_s \quad A_t = \int_0^t \mu(s,X_s)\,ds$

Theorem: $dX_t = \mu(t,X_t)\,dt + \sigma(t,X_t)\,dW_t$

4) $F: [0,T] \times \mathbb{R} \to \mathbb{R}$, $F \in \mathcal{C}^{1,2}([0,T]\times\mathbb{R})$

$$F(t,X_t) = F(0,X_0) + \int_0^t F_t(X_s)\,ds + \int_0^t F_x(X_s)\,\mu(s,X_s)\,ds + \int_0^t F_x(X_s)\,\sigma(s,X_s)\,dW_s + \frac{1}{2}\int_0^t F_{xx}(X_s)\,\sigma^2(s,X_s)\,ds$$

Proof: Take **3)** and replace dX_t with $\mu(t,X_t)\,dt + \sigma(t,X_t)\,dW_t$ and obsere $d[X]_t = d[\sigma W]_t = \sigma^2 d\underbrace{[W]_s}_{s}$

Example: $dS_t = \underbrace{S_t \mu}_{=\mu(t,S_t)} dt + \underbrace{S_t \sigma}_{=\sigma(t,S_t)} dW_t$ (geom. Brownian motion)

Compute $\ln(S_t)$: $f(x) = \ln(x)$, $f'(x) = \frac{1}{x}$, $f''(x) = -\frac{1}{x^2}$

$$\ln(S_t) = \ln(\underbrace{S_0}_{=1}) + \int_0^t \frac{1}{S_s}\underbrace{dS_s}_{=\mu S_s ds + \sigma S_s dW_s} + \frac{1}{2}\int_0^t -\frac{1}{S_s^2}\underbrace{d[S]_s}_{=d[S\sigma W]_s = S_s^2\sigma^2 d[W]_s = S_s^2\sigma^2 ds}$$

$$= \int_0^t \mu - \frac{\sigma^2}{2}\,ds + \int_0^t \sigma\,dW_s$$

$$= \left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W_t$$

$$\Leftrightarrow \quad S_t = \exp\left(\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W_t\right)$$

Feynman-Kac Formula:

Functions on $\mathbb{R}$: $\mu(x), \sigma(x), r(x)$ and $\phi(x)$ given.

Suppose $F(t,x)$ that solves the terminal value problem:

$$\frac{\partial F}{\partial t}(t,x) + \mu(x)\frac{\partial F}{\partial x}(t,x) + \frac{1}{2}\sigma^2(x)\frac{\partial^2 F}{\partial x^2}(t,x) = r(x)\,F(t,x) \quad \text{(partial differential equation)} \quad (2)$$

$$F(T,x) = \phi(x) \quad \text{(terminal value)}$$

Goal: represent F in terms of some Ito process

Consider SDE: $dX_t = \mu(X_t)\,dt + \sigma(X_t)\,dW_t, \quad X_{t_0} = x.$

Theorem: Feynman Kac

If F is suff. integrable (e.g. bounded), it holds for $t_0 \le T$ that

$$F(t_0, X_{t_0}) = \mathbb{E}_{X_{t_0}}\Big(\underbrace{\exp\Big(-\int_0^{T-t_0} r(X_s)\,ds\Big)}_{\text{discount}}\ \underbrace{\phi(X_{T-t_0})}_{\text{terminal value}}\Big) \qquad \mathbf{(2)}$$

How to use Feynman-Kac formula?

→ Compute **(2)** with e.g. Monte Carlo

→ Solve PDE **(1)** numerically to compute expectation **(2)**

Example: $f_t(t,x) + \mu f_x + \frac{1}{2}\sigma^2 f_{xx} = rf$ and $f(T,x) = x$

$$dX_t = \mu\,dt + \sigma\,dW_t$$
$$X_t = \mu t + \sigma W_t \sim \mathcal{N}(\mu t, \sigma^2 t) \qquad \text{(arithm. BM)}$$

$$f(0,X_0) = \mathbb{E}_{X_0}\Big[\exp\Big(-\int_0^T r\,ds\Big)\ X_T\Big] = e^{-rT}\,\mathbb{E}[X_T] = e^{-rT}\mu T$$

Recap last lecture

Itô-formula for Ito processes: $dX_t = \mu(t,X_t)\,dt + \sigma(t,X_t)\,dW_t$

$$F(t,X_t) = F(0,X_0) + \int_0^t F_t(s,X_s)\,ds + \int_0^t F_x(s,X_s)\,dX_s + \frac{1}{2}\int_0^t F_{xx}(s,X_s)\,d[X]_s$$

$$= F(0,X_0) + \int_0^t F_t(s,X_s)\,ds + \int_0^t F_x(s,X_s)\big(\mu(s,X_s)\,ds + \sigma(s,X_s)\,dW_s\big) + \frac{1}{2}\int_0^t F_{xx}(s,X_s)\,\sigma^2(s,X_s)\,\underbrace{d[W]_s}_{ds}$$

$$= F(0,X_0) + \underbrace{\int_0^t \Big(F_t(s,X_s) + \mu(s,X_s)F_x(s,X_s) + \frac{1}{2}\sigma^2(s,X_s)F_{xx}(s,X_s)\Big)\,ds}_{\substack{\text{Compare with Feynman-Kac} \\ \textbf{FV-process}}} + \underbrace{\int_0^t \sigma(s,X_s)\,dW_s}_{\textbf{local martingale}}$$

Solving SDEs: geometric Brownian motion

Solve: $dS_t = \mu S_t\,dt + \sigma S_t\,dW_t$, μ, σ constants, $S_0 = x$, Goal: $S_t = ?$

Ansatz: $F(S_t) = \ln(S_t)$, $F'(S_t) = \frac{1}{S_t}$, $F''(S_t) = -\frac{1}{S_t^2}$

$$F(S_t) = \ln(S_t) = \ln(S_0) + \int_0^t \frac{1}{S_s}\,dS_s + \frac{1}{2}\int_0^t -\frac{1}{S_s^2}\,\underbrace{d[S]_s}_{=[\sigma S W]_s}$$

$$= \ln(x) + \int_0^t \frac{1}{\cancel{S_s}}\big(\mu \cancel{S_s}\,ds + \sigma \cancel{S_s}\,dW_s\big) - \frac{1}{2}\int_0^t \frac{1}{\cancel{S_s^2}}\,\sigma^2 \cancel{S_s^2}\,\underbrace{d[W]_s}_{s}$$

$$= \ln(x) + \int_0^t \mu - \frac{\sigma^2}{2}\,ds + \int_0^t \sigma\,W_s$$

$$= \ln(x) + \Big(\mu - \frac{\sigma^2}{2}\Big)t + \sigma W_t \quad \overset{\exp}{\Longleftrightarrow} \quad S_t = x\exp\Big(\Big(\mu - \frac{\sigma^2}{2}\Big)t + \sigma W_t\Big)$$

Feynman Kac: Assume functions: $\mu(x), \sigma(x), r(x), \Phi(x)$

$$\underbrace{F_t(t,X) + \mu(X)\, F_x(t,X) + \frac{1}{2}\sigma^2(X)\, F_{xx}(t,X)}_{\text{Compare with Itô}} = r(x)\, F(t,x) \qquad \text{and} \qquad F(T,x) = \Phi(x)$$

Assume for simplicity $r(x) = 0$. With Itô: FV-part $= 0$ $\Rightarrow$ $F(t, X_t)$ is local martingale

$$\underbrace{F(t_0, X_{t_0})}_{\substack{\text{discounted}\\ \text{expected payoff}\\ \hat{=}\text{ option price}}} = \mathbb{E}_{\underbrace{X_{t_0}}} \Big[\exp\big(-\int_{t_0}^{T} \underbrace{r(X_s)}_{\text{discount rate}}\big)\; \underbrace{\Phi(X_T)}_{\text{payoff option}} \Big] \qquad \text{for } dX_t = \mu(X_t)\, dt + \sigma(X_t)\, dW_t$$

indicates cond. expectation

$\Rightarrow$ cond. exp. process defined as $X_t = \mathbb{E}[X_T | \mathcal{F}_t]$ is martingale for r.v. X_T.

$= \mathbb{E}_{X_t}[X_T]$

Example:

1) $F_t(t,X) + \mu X\, F_x(t,x) + \frac{\sigma^2}{2} x\, F_{xx}(t,x) = 0$, $X_0 = x$, $F(T,x) = X$

$dX_t = \mu X_t\, dt + \sigma X_t\, dW_t$ (geom. BM) , $X_0 = x$

$t_0 = 0$ $\quad F(0, X_0) = \mathbb{E}_x[\underbrace{\Phi(X_T)}_{X_T}] = \mathbb{E}[\underbrace{X_T}_{\text{geom. BM}}] = \mathbb{E}[x \exp((\mu - \frac{\sigma^2}{2})T + \sigma W_T)] = x \exp((\mu - \frac{\sigma^2}{2})T + \frac{\sigma^2}{2}T) = x e^{\mu T}$

2) $F_t(t,X) + \mu X\, F_x(t,x) + \frac{\sigma^2}{2} x\, F_{xx}(t,x) = r\, F(t,x)$ $\qquad F(T,x) = X$

$F(0, X_0) = \mathbb{E}[e^{-rT} X_T | X_0] = x_0 e^{(\mu - r)T}$ $\quad \Big(\overset{\text{for } \mu = r}{=} X_0 \quad (\text{if } S_t = X_t e^{-rt} \text{ martingale} \Rightarrow \mu = r) \Big)$

<u>Note</u>: For $\Phi(x) = (x - K)^+$ pricing formula for a call option

Exercise: Compute $S_t = e^{-rt} X_t$ where X_t is a geom. BM. and find a condition such that S_t is a local martingale! (Compare with Feynman-Kac)

$D_t = e^{-rt}$ (discount)

$dD_t = -r e^{-rt}\, dt$, $\qquad dX_t = \mu X_t\, dt + \sigma X_t\, dW_t \qquad X_0 = x_0$

$$S_t = D_t X_t \overset{\text{Ito Product formula}}{=} D_0 X_0 + \int_0^t D_s\, dX_s + \int_0^t X_s\, dD_s + \overbrace{[\underbrace{D}_{FV}, X]_t}^{=0}$$

$$= x_0 + \int_0^t e^{-rs} (\mu X_s\, ds + \sigma X_s\, dW_s) + \int_0^t X_s (-r e^{-rs})\, ds$$

$$= x_0 + \int_0^t e^{-rs} X_s (\mu - r)\, ds + \int_0^t e^{-rs} X_s\, \sigma\, dW_s$$

$$= x_0 + \int_0^t S_s \underbrace{(\mu - r)}\, ds + \int_0^t S_s\, \sigma\, dW_s \quad \leadsto \text{ again geom. BM}$$

S local martingale for $\mu = r$

The Black-Scholes Model

Assets: We consider

1) money market account B with $B_t = \exp(rt)$ for some $r > 0$ **(riskless)**

2) stock price S **(risky)**

Stock price dynamics: Filtered probability space $(\Omega, \mathcal{F}, P)$, $\{\mathcal{F}_t\}$ supports BM W_t.

Possible models for S (stock price):

1) arithm. BM (**Bachelier**): $S_t = S_0 + \underbrace{\mu t}_{\text{"trend"}} + \underbrace{\sigma W_t}_{\text{"noise"}}$, constants $\sigma, \mu > 0$

$S_t \sim \mathcal{N}(S_0 + \mu t, \sigma^2 t)$ => negative stock price

2) geometric BM (**Black Scholes Model**)

$$\boxed{S_t = S_0 \exp\left(\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W_t\right)}$$

$S_t \sim \log\text{-}\mathcal{N}\left(\left(\mu - \frac{\sigma^2}{2}\right)t + \ln(S_0), \sigma^2 t\right)$ => non-negative

Properties:

1) S solves SDE: $\boxed{dS_t = \mu S_t\, dt + \sigma S_t\, dW_t}$

Intuitive interpretation: $\underbrace{\frac{\overbrace{dS_t}^{\text{small change in } S}}{S_t}}_{\text{relative change}} = \mu \underbrace{dt}_{\text{small change in time}} + \sigma \underbrace{dW_t}_{\text{small change in } W} \iff \underbrace{\frac{\overbrace{S_{t+h} - S_t}^{dS_t}}{S_t}}_{\text{relative change}} = \mu \underbrace{(t+h-t)}_{dt = h} + \sigma \underbrace{(W_{t+h} - W_t)}_{dW_t} \sim \underbrace{\mu \cdot h}_{\text{"linear trend"}} + \sigma \underbrace{W_h}_{\substack{\varepsilon \sim \mathcal{N}(0,h) \\ \text{"noise"}}}$

2) Log-returns: $\ln(S_{t+h}) - \ln(S_t) = \sigma(W_{t+h} - W_t) + \left(\mu - \frac{\sigma^2}{2}\right)h \sim \mathcal{N}\left(\left(\mu - \frac{\sigma^2}{2}\right)h, \sigma^2 h\right)$

volatility σ is instantaneous variance of log-returns

Observe: (non-overlapping) log-returns are independent!

R-Code: trajectories

Advantages: (of BS-model)

1) Geom. Brownian motion gives a reasonable (but not perfect) description for asset price data.

2) Geom. BM allows for explicit pricing formulae for a relatively large class of derivatives.

3) Black Scholes model is quite robust as a model for hedging derivatives:

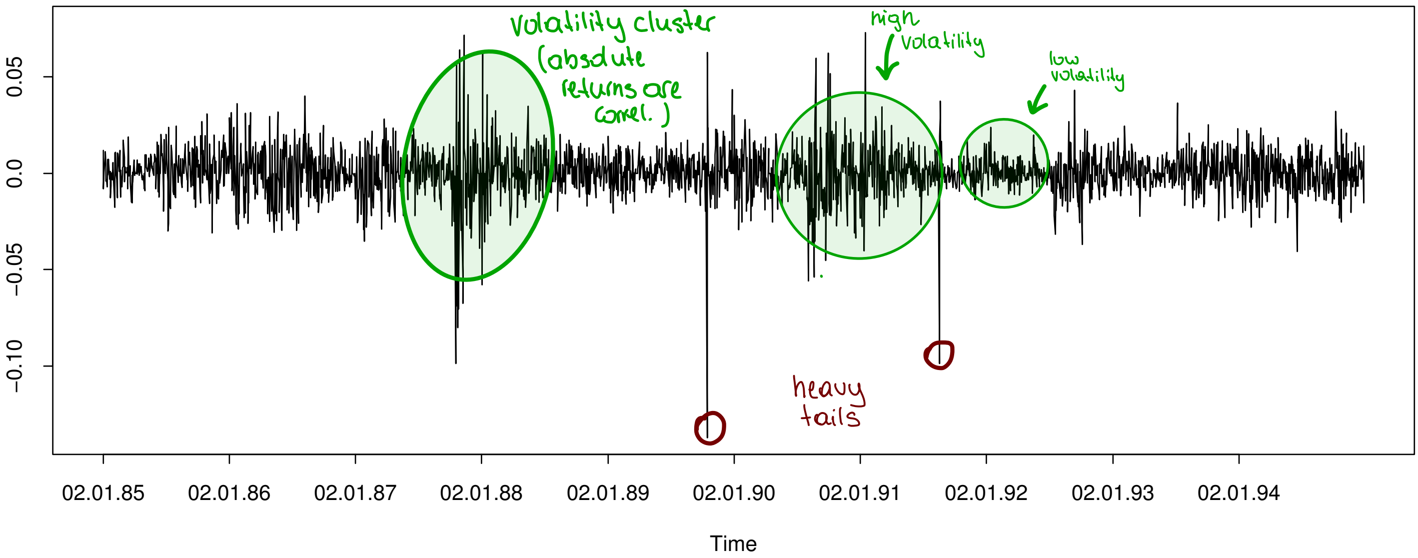
→ If real asset prices are not too different from geom. BM hedging strategies computed with BS model perform reasonably well.

Disadvantages:

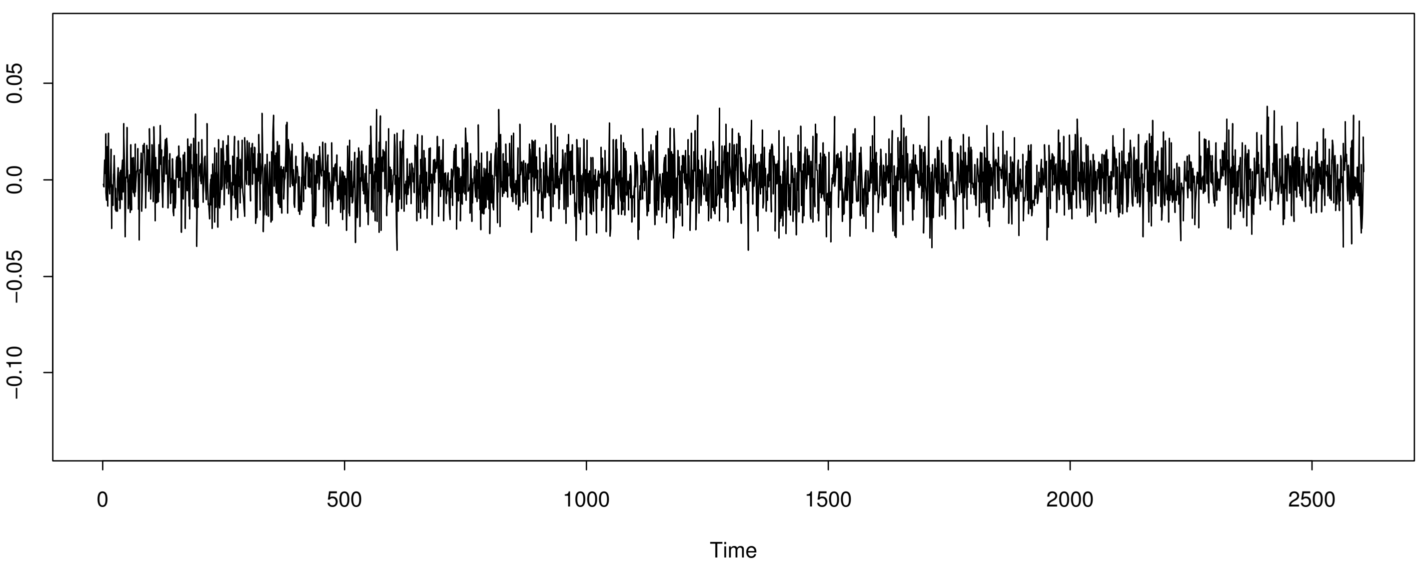
Financial data show strong evidence against normal distributed log-returns

- Returns are not iid (but corr. is low)
- absolute returns highly correlated
- Volatility appears to change randomly with time
- Returns are heavy tailed (e.g. Student-t-distr.)

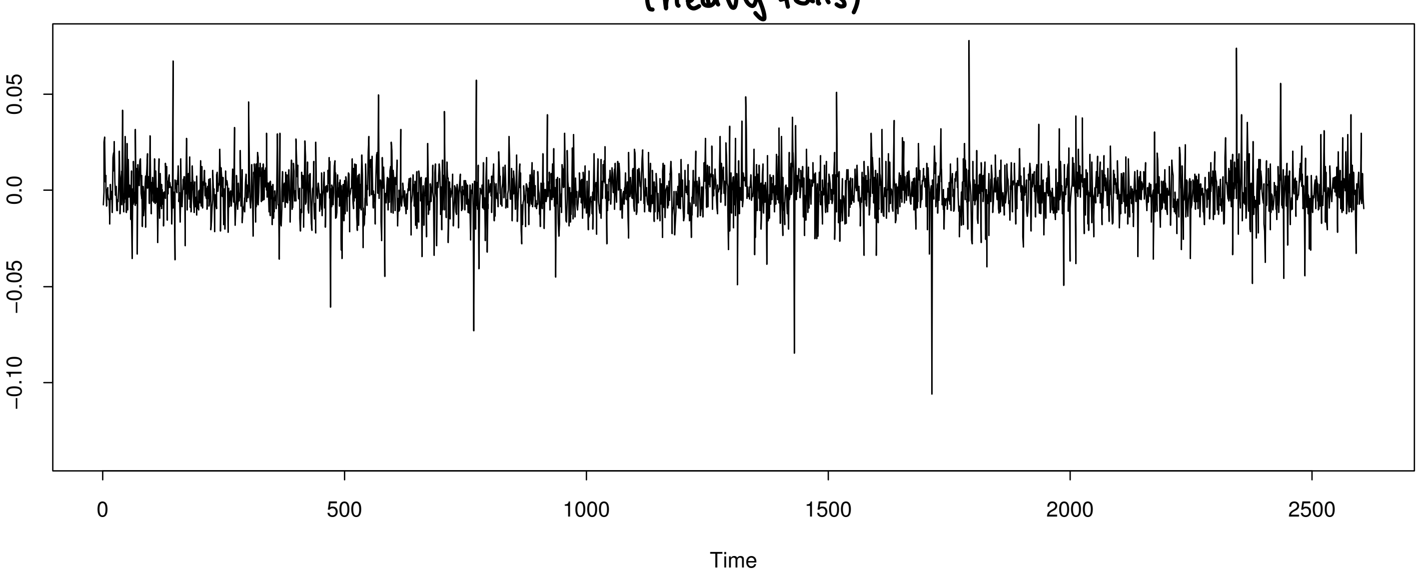
DAX (log-returns)

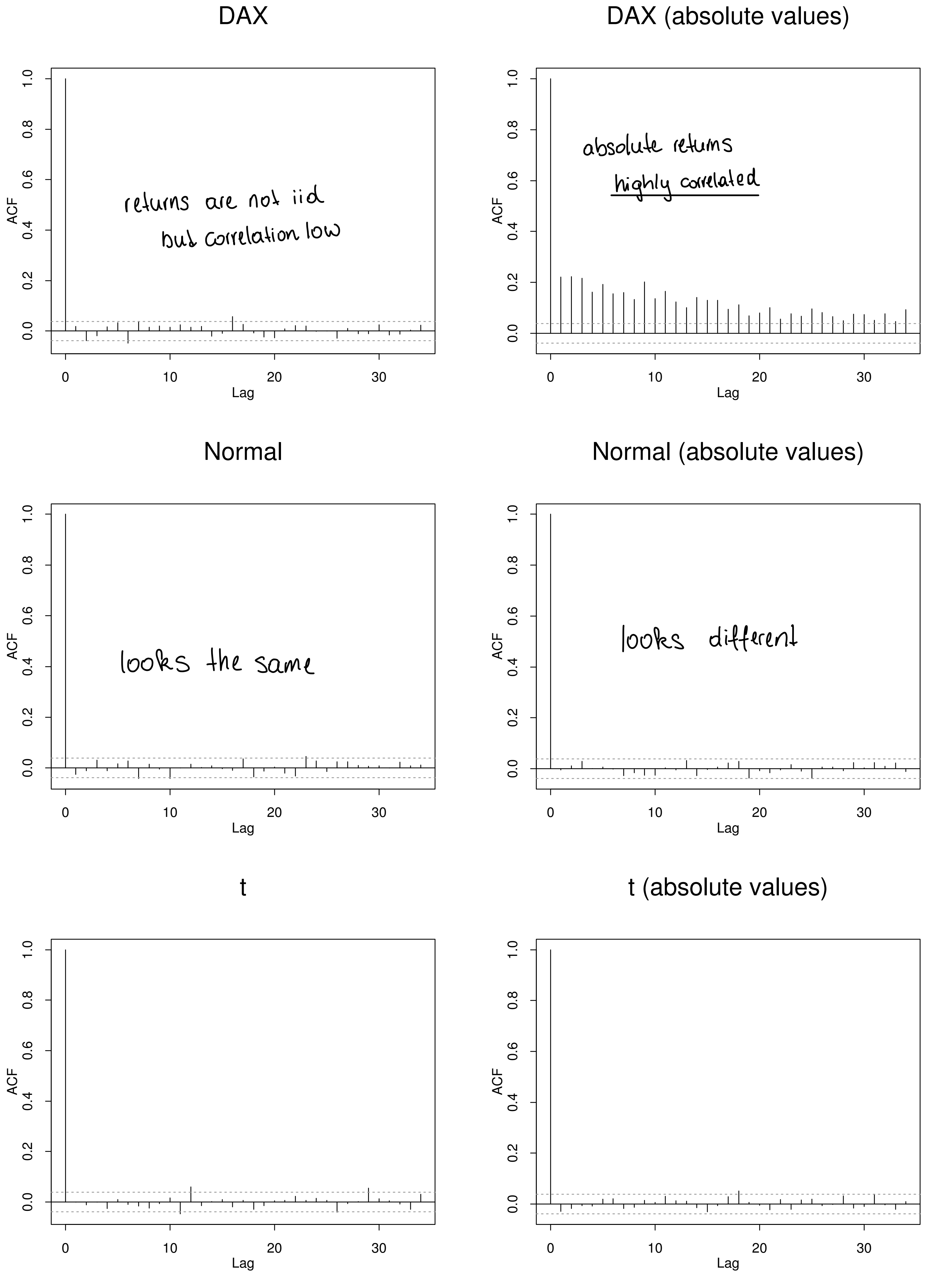


Normal



Student t
(heavy tails)




DAX
returns are not iid
but correlation low
DAX (absolute values)
absolute returns
highly correlated
Normal
looks the same
Normal (absolute values)
looks different
t
t (absolute values)
ACF
Lag
0
10
20
30
0.0
0.2
0.4
0.6
0.8
1.0

Pricing and Hedging of Terminal Value Claims

Pricing of a terminal value claim with payoff $H = h(S_T)$ (e.g. Call $h(S_T) = (S_T - K)^+$)

Goal: Find a dynamic replicating strategy for the claim

$\rightsquigarrow$ use strategy for pricing and hedging

Basic notation:

- A **Markov strategy** at time t is a given pair of smooth functions

$$\big(\underbrace{\phi(t,S_t)}_{\substack{\#\text{ stocks}\\ \text{risky}}},\ \underbrace{\eta(t,S_t)}_{\substack{\#\text{units money market}\\ \text{riskless}}}\big)$$

Markov: only depends on current value at time t

- The value at time t of the strategy is

$$\underbrace{V(t,S_t)}_{\substack{\text{value portfolio}\\ \text{consisting of}\\ \text{bank account and}\\ \text{stocks}}} = \underbrace{S_t}_{\substack{\text{stock}\\ \text{price}}} * \underbrace{\phi(t,S_t)}_{\#\text{ stocks}} + \underbrace{B_t}_{\substack{\text{value}\\ \text{bank account}\\ \text{or}\\ \text{riskless bond}}} * \underbrace{\eta(t,S_t)}_{\#\text{units at bank account}}$$

Note that $\eta(t,S_t) := (V(t,S) - S\phi(t,S))/B_t$ such that the portfolio can be described via V and ϕ.

Selffinancing strategies and gains from trade

Motivation: Markov trading strategy (ϕ, η) and partition $|\tau_n| \to 0$

Define piece wise constant approximations to (ϕ, η)

$$\phi_t^n(\omega) = \sum_{t_i \in \tau_n} \overset{(*)}{\phi}(t_{i-1}, S_{t_{i-1}}(\omega))\, 1_{(t_{i-1}, t_i]}(t)$$

$$\eta_t^n(\omega) = \sum_{t_i \in \tau_n} \eta(t_{i-1}, S_{t_{i-1}}(\omega))\, 1_{(t_{i-1}, t_i]}(t)$$

and let $V_t^n = \phi_t^n S_t + \eta_t^n B_t$. Discrete time finance: piecewise constant strategy is self-financing

$$\Leftrightarrow \forall t_i \in \tau_n \quad V_{t_i}^n = \underbrace{V_0}_{\substack{\text{initial portfolio}\\ \text{value}}} + \underbrace{G_{t_i}^n}_{\substack{\text{gains from}\\ \text{trade}}} \quad \text{where} \quad G_{t_i}^n = \sum_{j=1}^{i} \Big(\underbrace{\phi_{t_j}^n}_{\substack{\#\text{ stocks}\\ \text{in portfolio invested}\\ \text{at time } t_{j-1}.\ (*)}} \underbrace{(S_{t_j} - S_{t_{j-1}})}_{\substack{\text{change in stock}\\ \text{value in } (t_{j-1}, t_j]}} + \underbrace{\eta_{t_j}^n}_{\substack{\#\text{ bank}\\ \text{account}}} \underbrace{(B_{t_j} - B_{t_{j-1}})}_{\substack{\text{change bank acc.}\\ \text{value in } (t_{j-1}, t_j]}} \Big)$$

Selffinancing means: do NOT consume profits
do NOT compensate losses

Definition Ito-Integral:

$$\sum_{j=1}^{i} \phi_{t_j}^{n} (S_{t_j} - S_{t_{j-1}}) \xrightarrow{n \to \infty} \int_0^{t_i} \phi(s, S_s)\, dS_s$$

$$\sum_{j=1}^{i} \eta_{t_j}^{n} (B_{t_j} - B_{t_{j-1}}) \xrightarrow{n \to \infty} \int_0^{t_i} \eta(s, S_s)\, dS_s$$

Definition: Given a Markov-Strategy $(\phi(t, S_t^1), \eta(t, S_t^1))$ induced by smooth functions $\phi, \eta: [0,T] \times \mathbb{R}^+ \to \mathbb{R}$

(i) Gains from trade

$$G_t = \int_0^t \underbrace{\phi(s, S_s)}_{\substack{\#\text{ stocks} \\ \text{changes dynam.} \\ \text{for all } 0 \le s \le t}} \underbrace{dS_s}_{\substack{\text{change stock} \\ \text{price}}} + \int_0^t \underbrace{\eta(s, S_s)}_{\substack{\#\text{ units} \\ \text{bank account}}} \underbrace{dB_s}_{\substack{\text{change bank} \\ \text{account}}}$$

(ii) Selffinancing (ϕ, η): $\quad V(t, S_t) = \underbrace{V(0, S_0)}_{\substack{\text{initial} \\ \text{value}}} + G_t$

Note: $B_t = e^{rt}$, $dB_t = \overbrace{r \underbrace{e^{rt}}_{B_t}}^{(e^{rt})'} dt = r B_t\, dt$

$$G_t = \int_0^t \phi(s, S_s)\, dS_s + \int_0^t \eta(s, S_s)\, r B_s\, ds$$

Definition: Consider terminal value claim with payoff $h(S_T)$.

A selffinancing strategy is a **replicating strategy** for the claim if $\underbrace{V(T,S)}_{\substack{\text{portfolio value} \\ \text{(dep. on repl. str.)}}} = \underbrace{h(S)}_{\text{payoff}}$ for all $S > 0$.

In that case $V(t, S_t)$ is the fair price of the claim at time t.

Theorem: Let $V: [0,T] \times \mathbb{R}^+ \to \mathbb{R}$ be a continuous function which solves the PDE

$$V_t(t,S) + \tfrac{1}{2}\sigma^2 S^2 V_{SS}(t,S) + r S V_S(t,S) = r V(t,S) \qquad (t,S) \in [0,T) \times \mathbb{R}^+$$

Then the hedging strategy with stock position $\phi(t,S) = V_S(t,S)$ and value $V(t,S)$ is self-financing.

If V satisfies $V(T,S) = h(S)$, the strategy replicates the terminal value claim with payoff $h(S_T)$ and fair price equals $V(t, S_t)$.

Proof of Theorem:

1) Quadratic variation of geometric BM: $S_t = S_0 + \underbrace{\int_0^t \sigma S_s \, dW_s}_{=: M_t \text{ (loc. mg.)}} + \underbrace{\int_0^t r \, S_s \, ds}_{=: A_t \text{ (FV)}}$

$$[S]_t = [M_t] = \int_0^t \sigma^2 S_s^2 \underbrace{d[W]_s}_{ds}$$

2) Apply Ito on $V(t, S_t)$:

for simplicity: no arguments (s, S_s)

$$V(t,S_t) = V(0,S_0) + \int_0^t V_t \, ds + \int_0^t V_S \, dS_s + \frac{1}{2}\int_0^t V_{SS} \underbrace{d[S]_s}_{\sigma^2 S_s^2 ds}$$

$$= V(0,S_0) + \int_0^t \underbrace{V_S}_{=\emptyset(s,S_s) \text{ hedging strat. in theorem}} dS_s + \int_0^t \underbrace{\left(V_t + \tfrac{1}{2}\sigma^2 S_s^2 V_{SS}\right)}_{= rV - r\underbrace{V_S}_{\emptyset} S_s \;\} \text{ from PDE in the theorem!}} ds$$

$$= V(0,S_0) + \int_0^t \emptyset(s,S_s) \, dS_s + \int_0^t \underbrace{r\left(V - \emptyset S_s\right)}_{r \cdot B_s \cdot \eta(s,S_s)} ds \quad \text{as } \eta(t,S_t) := \left(V(t,S) - S\emptyset(t,S)\right)/B_t$$

$$= V(0,S_0) + \int_0^t \emptyset(s,S_s) \, dS_s + \int_0^t \eta(s,S_s) \underbrace{dB_s}_{= rB_s ds \text{ (Recall)}}$$

$\Rightarrow$ $(\emptyset, \eta)$ selffinancing

Second claim: Compare with Feynman-Kac: $\mu(t,S) = r \cdot S$, $\sigma(t,S) = \sigma S$ (geom. BM)

$dS_t = r S_t \, dt + \sigma dW_t$, e.g. $h(S_T) = (S_T - K)^+$ (Call Option)

$$\underbrace{V(0,S_0)}_{\text{fair price}} = \mathbb{E}_{S_0}\left[e^{-rT} h(S_T)\right] = \mathbb{E}\left[e^{-rT} h\left(S_0 \exp\left(\underbrace{(r - \tfrac{\sigma^2}{2})T + \sigma W_T}_{\mathcal{N}((r-\frac{\sigma^2}{2})T, \, \sigma^2 T)}\right)\right)\right]$$

$\rightsquigarrow$ Compute with MC or explicitly

Theorem: (risk-neutral pricing)

It holds that $V(t,S) = \mathbb{E}_S\left[e^{-r(T-t)} h(S_{T-t})\right]$

where S solves SDE $dS_t = rS_t \, dt + \sigma S_t W_t$

risk neutral pricing formula in continuous time

Stock has drift r (instead of μ) such that $e^{-rt} S_t$ is a martingale!

Compute the price of a European Call Option

$$V(t,S) = \mathbb{E}_S\left[e^{-r(T-t)} (\underbrace{S_T} - K)^+\right]$$

expectation is tricky to compute because of positive part!

where $S_T \sim \log\text{-}N\left(\ln(S_0) + (\mu - \tfrac{\sigma^2}{2})T, \, \sigma^2 T\right)$

Two possibilities: (to compute $V(t,S)$ for Call)

1) Solve PDE with $h(S) = (S-K)^+$ (Reduction to heat equation)

2) Compute risk neutral expectation.

Theorem (Black Scholes formula)

The price of a European call with strike K and maturity date T in the Black-Scholes model with volatility σ and interest rate r equals

$$C_{BS}(t, S; \sigma, r, K, T) := S \underbrace{N}_{\text{distribution function stand. Normalv.}}(d_1) - e^{-r(T-t)} K \underbrace{N}(d_2)$$

$$\text{with} \quad d_1 = \frac{\ln(S/K) + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}} \quad \text{and} \quad d_2 = d_1 - \sigma\sqrt{T-t}$$

European Put: Put-Call parity for price C_t and P_t of European Call and Put:

$$C_t + e^{-r(T-t)}K = S_t + P_t$$

This gives: $P_t = -S_t N(-d_1) + Ke^{-r(T-t)} N(-d_2)$

Delta of an option: The Delta of an option is the derivative wrt. the price of the underlying.

In Black-Scholes Model: $\Delta_C = \frac{\partial C}{\partial S} = N(d_1)$ and $\Delta_P = \frac{\partial P}{\partial S} = \Delta_C - 1 = -N(d_1)$

Hedging: The Delta is relevant for so-called **Delta-hedging**

- The hedge portfolio for a call:

  $\frac{\partial}{\partial S} C_{BS} = N(d_1)$ units of S (stock)

  $\frac{(C_{BS}(t,S) - N(d_1)S)}{e^{rt}} = -e^{-rT} KN(d_2)$ units of B (Bank account)

- For a put:

  $\frac{\partial}{\partial S} P_{BS} = -N(-d_1)$ units S

  $\frac{(P_{BS}(t,S) - (1-N(d_1))S)}{e^{rt}} = e^{-rT} KN(-d_2)$ units B

Proof of BS-formula:

$$C_{BS}(t,S;\sigma,r,K,T) := S\,N(d_1) - e^{-r(T-t)} K\,N(d_2)$$

with $d_1 = \dfrac{(\ln S)/K + (r+\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$ and $d_2 = d_1 - \sigma\sqrt{T-t}$

1) Solve PDE: apprach via heatequation; Proof sketch

Lemma: Define $\tau(t) = \sigma^2(T-t)$ and $z(t,S) = \ln S + (r-\frac{\sigma^2}{2})(T-t)$

Denote by $u(t,z): [0,T]\times\mathbb{R}\to\mathbb{R}$ the solution of $u_t = \frac{1}{2}u_{zz}$ (heat equation)

with $u(0,z) = (e^z - K)^+$. Then $C(t,S) = e^{-r(T-t)}u(\tau(t), z(t,S))$

solves the terminal value problem for the price of a European Call.

Proof: Check terminal value: $C(T,S) = u(\underbrace{\tau(T)}_{=\sigma^2(T-T)}, \underbrace{z(T,S)}_{\ln(S)+(r-\frac{\sigma^2}{2})(T-T)}) = \underbrace{u(0,\ln S)}_{=(e^{\ln(S)}-K)^+} = (S-K)^+$ ✓

$$\frac{\partial C}{\partial t} = e^{-r(T-t)} r\cdot u + e^{-r(T-t)}\cdot u_\tau\cdot(-\sigma^2) + e^{-r(T-t)}\cdot u_z\,(-r+\tfrac{\sigma^2}{2}) = e^{-r(T-t)}\left(ru - \sigma^2 u_\tau + (\tfrac{\sigma^2}{2}-r)u_z\right)$$

$$\frac{\partial C}{\partial S} = e^{-r(T-t)}u_z\frac{1}{S}, \qquad \frac{\partial^2 C}{\partial S^2} = e^{-r(T-t)}u_z\left(\frac{-1}{S^2}\right) + e^{-r(T-t)}u_{zz}\frac{1}{S^2} = e^{-r(T-t)}\frac{1}{S^2}(u_{zz}-u_z)$$

Plug in: $C_t + \frac{1}{2}\sigma^2 S^2 C_s + rS\,C_{ss} = rC$

$\Rightarrow$ and we get: $\underbrace{u_t = u_{zz}}$

Solution of heat equation is well known:

$$u(\tau,z) = \frac{1}{\sqrt{2\pi\tau}}\int_{-\infty}^{\infty} u_0(x)\,e^{-\frac{(z-x)^2}{2\tau}}\,dx \quad , \quad u(0,z) = u_0(z)$$

$\Rightarrow$ Result follows after tedious calculations.

2) Compute expectation: Probabilistic approach

$$V(t,S) = e^{-r(T-t)}\underbrace{\mathbb{E}_S\left[(S_T-K)^+\right]}$$

Compute expectation!

$$S_T = S_0\exp\Big(\underbrace{(r-\tfrac{\sigma^2}{2})(T-t)+\sigma W_{T-t}}\Big) = \exp(z)$$

$$= z \sim \mathcal{N}\left(\ln(S_0)+(r-\tfrac{\sigma^2}{2})(T-t),\ \sigma^2(T-t)\right)$$

For simplicity $t=0$. Let

$$\alpha := \frac{e^{-rT}}{\sqrt{2\pi\sigma^2 T}}, \quad \mu := \ln(S_0) + \left(r - \frac{\sigma^2}{2}\right)T, \quad \tilde{\sigma} := \sigma\sqrt{T}$$

$$e^{-rT}\,\mathbb{E}\left[(e^{z} - K)^+\right] = \alpha \int_{-\infty}^{\infty} (e^x - K)^+ \exp\left(-\frac{(x-\mu)^2}{2\tilde{\sigma}^2}\right) dx \qquad (\text{integrand} \neq 0 \text{ for } e^x > K \Leftrightarrow x > \ln K)$$

$$= \underbrace{\alpha \int_{\ln(K)}^{\infty} e^x \exp\left(-\frac{(x-\mu)^2}{2\tilde{\sigma}^2}\right) dx}_{=:\,I_1} - \underbrace{\alpha \int_{\ln(K)}^{\infty} K \exp\left(-\frac{(x-\mu)^2}{2\tilde{\sigma}^2}\right) dx}_{=:\,I_2}$$

Compute I_1: Integrand is of form $\exp(\lambda(x))$, **Idea**: Transform to normal distr. density!

$$\lambda(x) = x - \frac{(x-\mu)^2}{2\tilde{\sigma}^2} = -\frac{-2\tilde{\sigma}^2 x + x^2 - 2\mu x + \mu^2}{2\tilde{\sigma}^2} = -\frac{(x-(\mu+\tilde{\sigma}))^2 + (\mu^2 - (\mu+\tilde{\sigma}^2)^2)}{2\tilde{\sigma}^2}$$

$$= -\frac{(x-(\ln S_0 + (r+\frac{\sigma^2}{2})T))^2}{2\sigma^2 T} + \underbrace{(\ln S_0 + rT)}_{= \frac{(\mu+\tilde{\sigma}^2)^2 - \mu^2}{2\tilde{\sigma}^2} = \mu + \frac{\tilde{\sigma}^2}{2} = \ln S_0 + rT}$$

Using $\alpha e^{\ln S_0 + rT} = \frac{1}{\sqrt{2\pi\sigma^2 T}} S_0$ we get

$$\alpha I_1 = \frac{S_0}{\sqrt{2\pi\sigma^2 T}} \int_{\ln K}^{\infty} \exp\left(-\frac{(x-(\ln S_0 + (r+\frac{\sigma^2}{2})T))^2}{2\sigma^2 T}\right) dx$$

Consider $\tilde{z} \sim \mathcal{N}(\ln S_0 + (r+\frac{\sigma^2}{2})T,\ \sigma^2 T) \Rightarrow \frac{\tilde{z} - \ln S_0 + (r+\frac{\sigma^2}{2})T}{\sigma\sqrt{T}} \sim \mathcal{N}(0,1)$

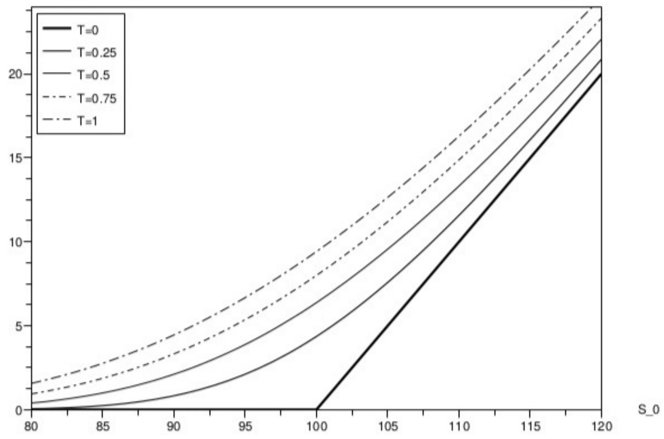
$$\alpha I_1 = S_0\, P(\tilde{z} > \ln(K)) = S_0\, P\left(\frac{\tilde{z} - \ln S_0 + (r+\frac{\sigma^2}{2})T}{\sigma\sqrt{T}} > \underbrace{\frac{\ln K - \ln S_0 + (r+\frac{\sigma^2}{2})T}{\sigma\sqrt{T}}}_{= -d_1}\right)$$

$$= S_0\,(1 - N(-d_1)) = S_0 N(d_1)$$

Similarly, $\alpha I_2 = e^{-rT} K N(d_2)$ (no quad. substitution necessary)

Example: $K = 100$, $r = 0.03$, $\sigma = 0.2$

Price of a Call Option



Price of a Put Option