Unit 3 Continuous Time Finance

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1 Exam questions

1.1 Explain the steps for defining the Ito-integral

The Ito-integral is defined as follows: $X_t = \int_0^t H_s dW_s :=_{n\to\infty}^{lim} \int_0^t H_{ns} dW_s$ The following steps shows how to arrive to this definition.

1. let $H_s = \sum_{i=1}^{n} Z_{i-1} * 1_{(t_{i-1},t_i]}(s)$ be an adapted, left-continuous step-process.
   $\Rightarrow X_t = \int_0^t H_s dW_s = \sum_{i=1}^{n} Z_{i-1} * \left(W_{t_i \cap t} - W_{t_{i-1} \cap t}\right)$ is well-defined integral, which is a random variable at the same time.
2. let $A_t = \int_0^t H_s^2 ds = \sum_{i=1}^{n} Z_{i-1}^2 * (t_i \cap t - t_{i-1} \cap t)$.
3. Now we want to show a kind of continuity:
   We use the Lemma 3.1: Let (H_t) be an adapted cadlag step process. If A_t is integrable, then X_t is a square integrable martingale and $X_t^2 - A_t$ is martingale, as well.
   So, we can use the "Isometric equality" and show
   $E\left[\left(\int_0^t H_s dW_s\right)^2\right] = E\left[\int_0^t H_s^2\right]$ which implies following:
   $H_n s \to 0 \Rightarrow \int_0^t H_s dW_s \to 0$ which is the desired continuity.
4. Now, if $E\left[\int_0^t H_s^2 ds\right] < \infty$ then for any Riemannian sequence of subdivisions of [0,t] the following holds:
   $H_{ns} := \sum_{i=1}^{n} H_{tn,i-1} * 1_{(t_{n,i-1},t_{n,i}]}(s) \to H_s$
5. So, now we arrive at the desired definition shown at the beginning:
   $X_t = \int_0^t H_s dW_s :=_{n\to\infty}^{lim} \int_0^t H_{ns} dW_s$

1.2 When is an Ito-integral a (square integrable) martingale?

Here we can use the Theorem 3.5, which specifies this property (it tells even more).
Theorem 3.5: The Ito-integral $X_t = \int_0^t H_s dW_s$ is defined for every adapted cadlad process (H_s) and is a local (square integrable) martingale. It is a square integrable martingale iff $E\left[\int_0^t H_s^2 ds\right] < \infty$. In this case the assertion of Lemma 3.1 is valied and the isometric equality holds.

1.3 Compare the properties of the Ito-integral and the integral w.r.t. a Poisson process

Ito-integral:$X_t = \int_0^t H_s dNs$ is:

- X_t is square integrable local martingale and if $E\left[\int_0^t H_s^2 ds\right] < \infty$, then even a martingale
- $X_t^2 - t$ is martingale

Stieltjes-Integrals w.r.t. a Poisson process

- $\int_0^t H_s dN_s$ is not a martingale
- $\int_0^t H_s d\left(N_s - \lambda_s\right)$ is a local martingale

1.4 When is the Wiener-integral a Levy-process?

Here, we have to check two properties, namely independency of the increments and stationarity of the increments.

- independent increments: see Proof 3
- stationary increments: the increments are stationary if their variance is proportional to t-s. Let's check it:
  $V\left(\int_0^t hdW\right) = \int_0^t h^2 ds = a{*}t = \int_0^t ads$. This is true only if $\int_0^t \left(h^2 - a\right) ds = 0 \Leftrightarrow h^2\left(s\right) = a \Rightarrow$ constant (under the assumption that h is positive or continuous).

1.5 What can we say about the increments of Ito-integral?

For Ito-integral:$X_t = \int_0^t H_s dW_s$, we can define the increment as follows:
if $t_1 < t_2 \Rightarrow$ the increment is $\int_{t_1}^{t_2} H_s dW_S$.

- Since the Ito-integral is martingale (if $E\left[\int_0^t H_s^2 ds\right] < \infty$) then the expected value of the increments has to be zero: $E\left[\int_{t_1}^{t_2} H_s dW_s | \mathcal{F}_s\right] = 0$
- Covariance for two increments where $s < t < u < v$ can be defined as $E\left[\int_s^t H_s dW_s * \int_u^v H_s dW_s\right]$ (recall that the expected value of Ito-integral is zero since Ito-integral is martingale (under some assumptions)). Therefore,
$E\left[\int_s^t H_s dW_s * \int_u^v H_s dW_s\right] = E\left[\int 1_{(s,t]} * 1_{(s,t]} * H_s^2 dW_s\right] = 0 \Rightarrow$ the increments are uncorrelated, BUT not necessarily independent. Based on the covariance, we cannot make any statement on independence, since Ito-integral is not Gaussian.

1.6 Explain why a square integrable Levy-process is a semimartingale.

Recall the definition of a semimartingale: A Semimartingale is an adapted cadlag process which is the sum of an FV-process and a local martingale."
Levy-process X_t can be rewritten in the following way: $X_t = \mu * t + M_t$, where M_t is a martingale and $\mu * t$ is a finite variation function.

2 Exam proofs

2.1 Prove Lemma 3.1

We have to prove three things:

1. $X_t = \int_0^t H_u dW_u$ is a martingale,
2. $X_t^2 - A_t$ is a martingale, and
3. X_t is square integrable

1. $X_t = \int_0^t H_u dW_u$ is a martingale ?
We have to show $[X_t - X_s | \{_s] = 0$
We can write the increment in the following way: $X_t - X_s = \sum_{i=1}^t Z_{i-1}\left(W_{t_i} - W_{t_{i-1}}\right) - \sum_{j=1}^s Z_{j-1}\left(W_{t_j} - W_{t_{j-1}}\right) = \sum_{i=s+1}^t Z_{i-1}\left(W_{t_i} - W_{t_{i-1}}\right)$
plugging in the conditional expectation: $E\left[X_t - X_s | \mathcal{F}_s\right] = E\left[\sum_{i=k+1}^t Z_{i-1}\left(W_{t_i} - W_{t_{i-1}}\right) | \mathcal{F}_s\right] = \sum_{i=k+1}^t E\left[Z_{i-1}\left(W_{t_i} - W_{t_{i-1}}\right) | \mathcal{F}_{t_k}\right] = \sum_{i=k+1}^t E\left[E\left[Z_{i-1}\left(W_{t_i} - W_{t_{i-1}}\right) | \mathcal{F}_{t_{i-1}}\right] \mathcal{F}_{t_k}\right] = \sum_{i=k+1}^n E\left[0 | \mathcal{F}_{t_k}\right] = 0$
Therefore X_t is a martingale.

2. $X_t^2 - A_t$ a martingale?
$E\left[X_t^2 - A_t|\mathcal{F}_s\right] = X_s^2 - A_s$??? We will show it in the following way:
$\left(X_t - X_s\right)^2 = X_t^2 - X_s^2 - 2X_s\left(X_t - X_s\right)$
$E\left[\left(X_t - X_s\right)^2|F_s\right] = E\left[X_t^2|\mathcal{F}_s\right] - E\left[X_s^2|\mathcal{F}_s\right] - 2X_sE\left[X_t - X_s|\mathcal{F}_s\right] = E\left[X_t^2|\mathcal{F}_s\right] - X_s^2$
If we can show that $E\left[\left(X_t - X_s\right)^2|F_s\right] = E\left[A_t - A_s|\mathcal{F}_s\right]$ then $X_t^2 - A_t$ is martingale.
$\left(X_t - X_s\right)^2 = \sum_{i=k+1}^n\left(Z_{i-1}\left(W_{t_i} - W_{t_{i-1}}\right)\right)^2 = \sum_{i=k+1}^n Z_{i-1}^2\left(W_{t_i} - W_{t_{i-1}}\right)^2 + \sum_{i\neq j}^n Z_{i-1}\left(W_{t_i} - W_{t_{i-1}}\right) * Z_{j-1}\left(W_{t_j} - W_{t_{j-1}}\right)$
Now we show that the last term is zero (assume $i < j$):
$E\left[Z_{i-1}\left(W_{t_i} - W_{t_{i-1}}\right) * Z_{j-1}\left(W_{t_j} - W_{t_{j-1}}\right)|\mathcal{F}_{t_k}\right] =$
$E\left[E\left[Z_{i-1}\left(W_{t_i} - W_{t_{i-1}}\right) * Z_{j-1}\left(W_{t_j} - W_{t_{j-1}}\right)\mathcal{F}_{t_{j-1}}\right]|\mathcal{F}_{t_k}\right] =$
$E\left[Z_{i-1}\left(W_{t_i} - W_{t_{i-1}}\right) * E\left[Z_{j-1}\left(W_{t_j} - W_{t_{j-1}}\right)|\mathcal{F}_[t_k]\right]\right] = 0$
Therefore: $E\left[\left(X_t - X_s\right)^2|\mathcal{F}_s\right] = E\left[\sum_{i=k+1}^n Z_{i-1}^2\left(W_{t_i} - W_{t_{i-1}}\right)^2|\mathcal{F}_{t_k}\right] = \sum_{i=k+1}^n E\left[E\left[Z_{i-1}^2\left(W_{t_i} - W_{t_{i-1}}\right)^2|\mathcal{F}_{ti-1}\right]|\mathcal{F}_{t_k}\right] = \sum_{i=k+1}^n E\left[Z_{i-1}^2\left(t_i - t_{i-1}\right)|\mathcal{F}_{t_k}\right] = E\left[\sum_{i=k+1}^n\left(Z_{i-1}^2\left(t_i - t_{i-1}\right)\right)|\mathcal{F}_{t_k}\right] = E\left[\int_s^t Z_u^2du|\mathcal{F}_s\right] = E\left[A_t - A_s|\mathcal{F}_s\right]$

2.2 Wiener integral has normal distribution

Wiener integral is defined as follows: $\int_0^t h_s dW_s$.
For h being a step function $h = \sum_{i=1}^n a_i * 1_{(t_{i-1},t_i]}$ we can define the Wiener integral X_t as follows: $X_t = \int_0^t h_s dW_s = \sum_{i=1}^n a_i * \left(W_{t_i} - W_{t_{i-1}}\right)$ which is a linear combination of normally distributed independent random variables and hence normal distributed.

Now, we will show it for any function h:
let $h_n = \sum_{i=1}^n a_i * 1_{(t_{i-1},t_i]} \Rightarrow \int_0^h h_n dW = \sum_{i=1}^n a_i * (W_{ti-1}, W_{ti})$ is linear combination of normaly distributed random variables $\Rightarrow$ normally distributed.
Now we define function h as a limit of functions h_n: $h = lim_{n\to\infty}h_n$
$X_t = \int_0^t hdW = lim_{n\to\infty}\int_0^h h_n dW$ and X_t is normally distributed as well.

2.3 Covariance of two Wiener integrals

$V\left(\int_0^t h_1dW + \int_0^t h_2dW\right) = V\left(\int_0^t h_1 + h_2dW\right) = \int_0^t\left(h_1 + h_2\right)^2 ds = \int_0^t h_1^2ds + \int_0^t h_2^2ds + 2\int_0^t h_1h_2ds$
We can write the first expression as:
$V\left(\int_0^t h_1dW\right) + V\left(\int_0^t h_2dW\right) + 2Cov\left(\int_0^t h_1dW, \int_0^t h_2dW\right)$
If we compare these two "results" we can show that:
$2Cov\left(\int_0^t h_1dW, \int_0^t h_2dW\right) = 2\int_0^t h_1h_2ds$

Therefore, $Cov\left(\int_0^t h_1 dW, \int_0^t h_2 dW\right) = \int_0^t h_1 h_2 ds$

2.4 Wiener integrals have independent increments

$X_t \int_0^t hdW$ Let's define two increments (where $s < t < u < v$)
$X_t - X_s = \int_s^t hdW = \int 1_{(s,t]} hdW$ and
$X_v - X_u = \int_u^v hdW = \int 1_{(u,v]} hdW$.
$cov\left(X_t - X_s, X_v - X_u\right) = \int 1_{(s,t]} * 1_{(u,v]} h^2 ds = 0$ (It is so, because (s,t] and (u,v] are disjoint intervals)
since X_t is normal distributed (see proof 2), we can conclude that the covariance of zero implies independence of the increments.

2.5 Find the covariance of two Ito-integrals

$V\left(\int_0^t HdW + \int_0^t GdW\right) = E\left[\left(\int_0^t H + GdW\right)^2\right] - \left(E\left[\int_0^t H + GdW\right]\right)^2 = E\left[\left(\int_0^t H + GdW\right)^2\right]$
Let us define $I_1(t)$ and $I_1(t)$
$I_1(t) = \int_0^t HdW$ $I_2(t) = \int_0^t GdW$
Now we can write the covariance of $\int_0^t HdW$ and $\int_0^t GdW$ as
$cov\left(I_1(t), I_2(t)\right) = E\left[I_1(t) * I_2(t)\right] - E\left[I_1(t)\right] * E\left[I_2(t)\right]$.
Since $I_1(t)$ and $I_v(t)$ are martingales, the last term in the previous equation cancels out and we get: $cov\left(I_1(t), I_2(t)\right) = E\left[I_1(t) * I_2(t)\right]$

Now we show that $I_1(t) * I_2(t) = \frac{(I_1(t)+I_2(t))^2 - I_1(t)^2 - I_2(t)^2}{2}$. Taking the expectation of this (since $cov\left(I_1(t), I_2(t)\right) = E\left[I_1(t) * I_2(t)\right]$) leads to:
$E\left[I_1(t) * I_2(t)\right] = \frac{E\left[(I_1(t)] + E[I_2(t))^2\right] - E\left[I_1(t)^2 - I_2(t)^2\right]}{2}$
Now we can plug in following expressions (using isometric equality):
$E\left[I_1(t)^2\right] = E\left[\left(\int_0^t HdW\right)^2\right] = E\left[\int_0^t H^2 ds\right] = \int_0^t E\left[H^2\right] ds$
$E\left[I_2(t)^2\right] = E\left[\left(\int_0^t GdW\right)^2\right] = E\left[\int_0^t G^2 ds\right] = \int_0^t E\left[G^2\right] ds$
$E\left[\left(I_1(t) + I_2(t)\right)^2\right] = E\left[\left(\int_0^t H + GdW\right)^2\right] = E\left[\int_0^t (H+G)^2 ds\right] = \int_0^t E\left[(H+G)^2\right] ds$
Plugging in the expression for covariance gives:
$cov\left(I_1(t), I_2(t)\right) = \frac{\int_0^t E\left[(H+G)^2\right] ds - E\left[\int_0^t G^2 ds\right] = \int_0^t E\left[G^2\right] ds - E\left[\int_0^t H^2 ds\right] = \int_0^t E\left[H^2\right] ds}{2} =$
$\int_0^t E\left[H_s G_s\right] ds$.

2.6 Show that a square integrable martingale which is continuous and FV must be constant

Let's define M_t as a continuous square integrable martingale.
Moreover, let us assume (w.l.o.g) $M_0 =$.
Since M_t is a finite variation process, we can use the integration by parts formula to write $M_t^2 - M_0^2 = \int_0^t M_s dMs + \int_0^t M_s dM_s \Rightarrow M_t^2 = 2\int_0^t M_s dM_s$

There exists following theorem:
Theorem: Suppose M is continuous FV square integrable martingale $\Rightarrow Y_t = \int_0^t M_s dM_s$ is a continuous martingale.
By the theorem we know that $M_t^2 = \int_0^t M_s dM_s$ is a continuous martingale.
$\Rightarrow E\left[M_t^2\right] = E\left[M_0^2\right] = 0.$
So, we also know that if $x \geq 0 a.s.$ and $E\left[X\right] = 0 \Rightarrow x = 0 a.s.$.
Therefore, M_t is constant.

2.7 Show that the representation of an Ito-process is uniquely determined

Suppose we have two representation for the Ito-process. We will show that they are identical.
$x_t = x_0 + \int_0^t a_s ds + \int_0^t b_s dW_s$
$x_t = x_0 + \int_0^t \overline{a_s} ds + \int_0^t \overline{b_s} dW_s$
Subtracting these two expressions gives:
$0 = \int_0^t \left(a - \overline{a_s}\right) ds + \int_0^t \left(b_s - \overline{b_s}\right) dW_s$
$\int_0^t \left(a - \overline{a_s}\right) ds = \int_0^t \left(\overline{b_s} - b\right) dW_s$
Now, the argumentation is following: the left-hand side is a FV-process. Therefore, the right-hand side is FV, continuous (because of the jump rule - W_s is continuous), square integrable martingale).
Let as write the right-hand side as $y_t = \int_0^t \left(\overline{b_s} - b_s\right) dW_s$ is continuous FV square-integrable martingale. Then by Theorem 3.12 $y_t = y_0 = 0$ for all t. Therefore, the left-hand side $\left(\int_0^t \left(a_s - \overline{a_s}\right) ds\right)$ is constant as well and $a = \overline{a}$

Now, we have to show that $b = \overline{b}$
$var\left(\int_0^t \left(\overline{b}_s b_s\right) dW_s\right) = E\left[\left(\int_0^t \left(\overline{b}_s - b_s\right) dW_s\right)^2 = \int_0^t E\left[\left(\overline{b}_s - b_s\right)^2\right] ds\right.$ (by isometric equality.
Since $y_t = 0 \Rightarrow Var\left(y_t\right) = 0$
$0 = \int_0^t E\left[\left(\overline{b}_s - b_s\right)^2\right] ds = 0.$
Therefore, $b_s = \overline{b}_s$.

2.8 Let M_t be a square integrable martingale. Show that there is at most one continuous FV process such that $M_t^2 - C_t$ is a square integrable martingale.

Suppose C_t and A_t are continuous FV processes s.t. $M_t^2 - C_t$ $M_t^2 - A_t$ are square integrable martingales.
$C_t = M_t^2 - mart.$
$A_t = M_t^2 - mart.$
Subtracting these two expressions gives: $C_t - A_t =$ square integrable martingale
Since the left-hand side is FV, the right-hand side is FV as well.
$\Rightarrow C_t - A_t$ is a continuous square integrable FV-martingale.
By theorem 3.12 $\Rightarrow C_t - A_t$ is constant.
Assume (w.l.o.g) that $A_0 = C_0 = 0$ then $C_t - A_t = C_0 - A_0 = 0 \Rightarrow C_t = A_t$.

3 Exam problems

3.1 Find expectation and variance and check the martingale property

1. $\int_0^t e^{2W_s} dW_s$
   First, we will check whether $A_t = \int_0^t \left(e^{2W_s}\right)^2 ds$ is integrable:
   $E\left[\int_0^t \left(e^{2W_s}\right)^2 ds\right] =$ using Fubini-Theorem $= \int_0^t E\left[\left(e^{2W_s}\right)^2\right] ds = \int_0^t E\left[e^{4W_s}\right] ds = \int_0^t e^{\frac{16s}{2}} = \frac{1}{8} * \left(e^{8t} - 1\right)$
   $\Rightarrow A_t$ is integrable $\Rightarrow X_t$ square integrable martingale
   $\Rightarrow E\left[X_t\right] = 0$
   variance of X_t: $V\left(X_t\right) = E\left[\left(\int_0^t e^{2W_s} dWs\right)^2\right] =$ by isometric equality $= E\left[\int_0^t e^{4W_s} ds\right] = \frac{1}{8} * \left(e^{8t} - 1\right)$

2. $X_t = \int_0^t 2^{W_s} dW_s$ First, we will check whether $A_t = \int_0^t \left(2^{W_s}\right)^2 ds$ is integrable:
   $E\left[\int_0^t \left(2^{W_s}\right)^2 ds\right] =$ using Fubini $= \int_0^t E\left[\left(2^{2W_s}\right)\right] ds$
   In order to get $e^{\cdots}$ expression in our expected value, we conduct following steps: $2 = e^{ln 2^{W_s}} = e^{W_s ln 2}$ plugging in the expression above, we get:
   $= \int_0^t E\left[\left(e^{W_s ln 2}\right)\right] ds = \int_0^t e^{\frac{4(ln 2)^2 s}{2}} ds = \int_0^t e^{2(ln 2)^2 s} ds$
   Let $c = 2\left(ln 2\right)^2$, then $\int_0^t e^{2(ln 2)^2 s} ds = \int_0^t e^{cs} ds = \frac{1}{c}\left(e^{ct} - 1\right)$
   $\Rightarrow X_t$ is a square integrable martingale.
   $E\left[X_t\right] = 0$
   variance: $V\left(X_t\right) = E\left[\left(\int_0^t 2^{w_s} dW_s\right)^2\right] =$ by isometric equality $= E\left[\int_0^t \left(2^{W_s}\right)^2 ds\right] = \frac{1}{c}\left(e^{ct} - 1\right)$

plugging c gives: $\frac{1}{2(ln2)^2}\left(e^{2(ln2)^2}-1\right)$

3. $X_t = \int_0^t W_s^2 dW_s$
   $X_t = \int_0^t W_s 4ds$ integrable?
   $E\left[\int_0^t W_s^4 ds\right] = \int_0^t E\left[W_s^4\right] ds = \int_0^t 3s^2 ds = \left(t^3\right)$
   $\Rightarrow X_t$ is square integrable martingale
   $\Rightarrow E\left[X_t\right] = 0$
   variance: $V\left(X_t\right) = E\left[\left(\int_0^t w_s^4 dW_s\right)^2\right] =$ by isometric equality $= E\left[\int_0^t W_s^4 ds\right] = t^3$

3.2 Find the covariance of

1. W_t and $\int_0^t s dW_s$
   we can define $W_t = \int_0^t 1 dW_s$. Therefore, the covariance is $cov\left(\int_0^t 1dW_s, \int_0^t s dW_s\right) = E\left[\int_0^t 1 * s ds\right] = \frac{t^2}{2}$
2. W_t and $\int_0^t w_s dW_s$
   $cov\left(\int_0^t 1dW_s, \int_0^t W_s dW_s\right) = E\left[\int_0^t 1 * W_s ds\right] =$ using Fubini-Theorem $= \int_0^t E\left[W_s\right] ds = 0$
   (compare with exam question 5. (What can we say about the increments of Ito-integral): the covariance is zero, BUT we cannot claim that there is no independence)

3.3 let $X_t = \int_0^t s dW_s$. Find expectation and variance and check the martingale property of:

1. $\int_0^t X_s dW_s$
   First, we will check the integrability of $\int_0^t X_s^2 ds$. $\Rightarrow E\left[\int_0^t X_s^2 ds\right] < \infty$

   Using Fubini-Theorem, we get $E\left[\int_0^t X_s^2 ds\right] = \int_0^t E\left[X_s^2\right] ds$

   for $E\left[X_s^2\right]$ we can show: $E\left[X_s^2\right] = E\left[\left({}_0^t s dW_s\right)^2\right] =$ by isometric equality (under the assumption that $\left(\int_0^t s^2 ds\right.$ integrable) $= E\left[\int_0^t s^2 dW_s\right] = E\left[\frac{s^3}{3}\right] = \frac{s^3}{3}$
   Now, we plug in this result for $E\left[X_s^2\right]$ and get:
   $\int_0^t E\left[X_s^2\right] ds = \int_0^t \frac{s^3}{3} ds = \frac{s^4}{12}$
   $\Rightarrow \int_0^t X_s^2 ds$ is integrable
   $\int_0^t X_s dW_s$ is square integrable martingale

$\Rightarrow E\left[\int_0^t X_s dW_s\right] = 0$

variance $V\left(\int_0^t X_s dW_s\right) = E\left[\left(\int_0^t X_s dW_s\right)^2\right] = E\left[\int_0^t X_s^2 ds\right]$ and we have already seen that this equals to $\frac{s^4}{12}$

2. $\int_0^t W_s dX_s =$ here we apply the associativity rule $= \int_0^t W_s * s dW_s$
Now, we will check the integrability of $A_t = \int_0^t (W_s * s)^2$:
$E\left[\int_0^t (W_s * s)^2 ds\right] =$ by Fubini $\int_0^t E\left[(W_s * s)^2\right] ds = \int_0^t s^2 E\left[(W_s)^2\right] ds = \int_0^t s^3 ds = \frac{t^4}{4}$
$\Rightarrow A_t = \int_0^t (W_s * s)^2$ is integrable
$\int_0^t W_s dX_s$ is a square integrable martingale.
$E\left[\int_0^t W_s dX_s\right] = 0$
variance: $V\left(\int_0^t W_s dX_s\right) = E\left[\left(\int_0^t W_s dX_s\right)^2\right] = \frac{t^4}{4}$ (We have already showed it when we checked the integrability of A_t).